

Pointwise Characterizations of Besov and Triebel-Lizorkin Spaces and Quasiconformal Mappings

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Abstract In this paper, the authors characterize, in terms of pointwise inequalities, the classical Besov spaces $\dot{B}_{p,q}^s$ and Triebel-Lizorkin spaces $\dot{F}_{p,q}^s$ for all $s \in (0, 1)$ and $p, q \in (n/(n+s), \infty]$, both in \mathbb{R}^n and in the metric measure spaces enjoying the doubling and reverse doubling properties. Applying this characterization, the authors prove that quasiconformal mappings preserve $\dot{F}_{n/s,q}^s$ on \mathbb{R}^n for all $s \in (0, 1)$ and $q \in (n/(n+s), \infty]$. A metric measure space version of the above morphism property is also established.

1 Introduction

We begin by recalling the metric definition of quasiconformal mappings and the definition of quasisymmetric mappings; see [29]. Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces and $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a homeomorphism. If there exists $H \in (0, \infty)$ such that for all $x \in \mathcal{X}$,

$$\limsup_{r \rightarrow 0} \frac{\sup\{d_{\mathcal{Y}}(f(x), f(y)) : d_{\mathcal{X}}(x, y) \leq r\}}{\inf\{d_{\mathcal{Y}}(f(x), f(y)) : d_{\mathcal{X}}(x, y) \geq r\}} \leq H,$$

then f is called *quasiconformal*. Moreover, if there exists a homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$ such that for all $a, b, x \in \mathcal{X}$ with $x \neq b$,

$$\frac{d_{\mathcal{Y}}(f(x), f(a))}{d_{\mathcal{Y}}(f(x), f(b))} \leq \eta\left(\frac{d_{\mathcal{X}}(x, a)}{d_{\mathcal{X}}(x, b)}\right),$$

then f is called η -*quasisymmetric*, and sometimes, simply, quasisymmetric. Every quasisymmetric mapping is quasiconformal, but the converse is always not true; see, for example, [16] and the references therein.

Let $n > 1$ and $\mathcal{X} = \mathcal{Y} = \mathbb{R}^n$ equipped with the usual Euclidean distance. Then quasiconformality is equivalent with quasisymmetry and further with the analytic conditions that the first order distributional partial derivatives of f are locally integrable and

$$|Df(x)|^n \leq KJ(x, f)$$

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almost everywhere (assuming that f is orientation preserving). The well-known result that the Sobolev space $\dot{W}^{1,n}$ is invariant under quasiconformal mappings on \mathbb{R}^n then comes as no surprise; see, for example, [19, Lemma 5.13]. By a function space being invariant under quasiconformal mappings we mean that both f and f^{-1} induce a bounded composition operator. Reimann [25] proved that also BMO is quasiconformally invariant by employing the reverse Hölder inequalities of Gehring [9] for the Jacobian of a quasiconformal mapping. Both the above two invariance properties essentially characterize quasiconformal mappings [25]. These results extend to the setting of Ahlfors regular metric spaces that support a suitable Poincaré inequality [16], [20]. There are some further function spaces whose quasiconformal invariance follows from the above results. First of all, the trace space of $\dot{W}^{1,n+1}(\mathbb{R}^{n+1})$ is the homogeneous Besov space $\dot{\mathcal{B}}_{n+1}^{n/(n+1)}(\mathbb{R}^n)$; see Section 4 for the definition. Because each quasiconformal mapping of \mathbb{R}^n onto itself extends to a quasiconformal mapping of \mathbb{R}^{n+1} onto itself [30], one concludes the invariance of $\dot{\mathcal{B}}_{n+1}^{n/(n+1)}(\mathbb{R}^n)$ with a bit of additional work. Further function spaces that are invariant under quasiconformal changes of variable are obtained using interpolation. For this, it is convenient to work with Triebel-Lizorkin spaces, whose definitions will be given in Section 3. Recall that $\text{BMO}(\mathbb{R}^n) = \dot{F}_{\infty,2}^0(\mathbb{R}^n)$, $\dot{W}^{1,n}(\mathbb{R}^n) = \dot{F}_{n,2}^1(\mathbb{R}^n)$, and $\dot{\mathcal{B}}_{n+1}^{n/(n+1)}(\mathbb{R}^n) = \dot{F}_{n+1,n+1}^{n/(n+1)}(\mathbb{R}^n)$. By interpolation, one concludes that also the Triebel-Lizorkin spaces $\dot{F}_{n/s,2}^s(\mathbb{R}^n)$ are invariant for all $s \in (0, 1)$ and so are $\dot{F}_{n/s,q}^s(\mathbb{R}^n)$ when $s \in (0, n/(n+1))$ and $q = 2n/(n - (n-1)s)$ or when $s \in (n/(n+1), 1)$ and $q = 2/((n-1)s - n + 2)$. Notice that above the allowable values of q satisfy $2 < q < n/s$.

Recently, Bourdon and Pajot [2] (see also [1]) proved a general result for quasisymmetric mappings, which, in the setting of \mathbb{R}^n , shows that the Triebel-Lizorkin space $\dot{F}_{n/s,n/s}^s(\mathbb{R}^n)$ is quasiconformally invariant, for each $s \in (0, 1)$. Notice that the norms of all the Triebel-Lizorkin spaces considered above are conformally invariant: invariant under translations, rotations and scalings of \mathbb{R}^n . It is then natural to inquire if all such Triebel-Lizorkin spaces are quasiconformally invariant.

Our first result shows that this is essentially the case.

Theorem 1.1. *Let $n \geq 2$, $s \in (0, 1)$ and $q \in (n/(n+s), \infty]$. Then $\dot{F}_{n/s,q}^s(\mathbb{R}^n)$ is invariant under quasiconformal mappings of \mathbb{R}^n .*

The assumption $q > n/(n+s)$ may well be superficial in Theorem 1.1, because of the way it appears in our estimates. Indeed, the proof of the above theorem is rather indirect: we establish the quasiconformal invariance of a full scale of spaces defined by means of pointwise inequalities initiated in the work of Hajlasz [11] and verify that, for most of the associated parameters, these spaces are Triebel-Lizorkin spaces. Let us introduce the necessary notation.

In what follows, we say that (\mathcal{X}, d, μ) is a *metric measure space* if d is a metric on \mathcal{X} and μ a regular Borel measure on \mathcal{X} such that all balls defined by d have finite and positive measures.

Definition 1.1. Let (\mathcal{X}, d, μ) be a metric measure space. Let $s \in (0, \infty)$ and u be a measurable function on \mathcal{X} . A sequence of nonnegative measurable functions, $\vec{g} \equiv \{g_k\}_{k \in \mathbb{Z}}$,

is called a *fractional s -Hajlasz gradient* of u if there exists $E \subset \mathcal{X}$ with $\mu(E) = 0$ such that for all $k \in \mathbb{Z}$ and $x, y \in \mathcal{X} \setminus E$ satisfying $2^{-k-1} \leq d(x, y) < 2^{-k}$,

$$(1.1) \quad |u(x) - u(y)| \leq [d(x, y)]^s [g_k(x) + g_k(y)].$$

Denote by $\mathbb{D}^s(u)$ the *collection of all fractional s -Hajlasz gradients of u* .

In fact, $\vec{g} \equiv \{g_k\}_{k \in \mathbb{Z}}$ above is not really a gradient. One should view it, in the Euclidean setting (at least when $g_k = g_j$ for all k, j), as a maximal function of the usual gradient. Relying on this concept we now introduce counterparts of Besov and Triebel-Lizorkin spaces. For simplicity, we only deal here with the case $p \in (0, \infty)$; the remaining case $p = \infty$ is given in Section 2. In what follows, for $p, q \in (0, \infty]$, we always write $\|\{g_j\}_{j \in \mathbb{Z}}\|_{\ell^q} \equiv \{\sum_{j \in \mathbb{Z}} |g_j|^q\}^{1/q}$ when $q < \infty$ and $\|\{g_j\}_{j \in \mathbb{Z}}\|_{\ell^\infty} \equiv \sup_{j \in \mathbb{Z}} |g_j|$,

$$\|\{g_j\}_{j \in \mathbb{Z}}\|_{L^p(\mathcal{X}, \ell^q)} \equiv \| \|\{g_j\}_{j \in \mathbb{Z}}\|_{\ell^q} \|_{L^p(\mathcal{X})}$$

and

$$\|\{g_j\}_{j \in \mathbb{Z}}\|_{\ell^q(L^p(\mathcal{X}))} \equiv \| \|\{g_j\}_{j \in \mathbb{Z}}\|_{L^p(\mathcal{X})} \|_{\ell^q}.$$

Definition 1.2. Let (\mathcal{X}, d, μ) be a metric measure space, $s, p \in (0, \infty)$ and $q \in (0, \infty]$.

(i) The *homogeneous Hajlasz-Triebel-Lizorkin space* $\dot{M}_{p,q}^s(\mathcal{X})$ is the space of all measurable functions u such that

$$\|u\|_{\dot{M}_{p,q}^s(\mathcal{X})} \equiv \inf_{\vec{g} \in \mathbb{D}^s(u)} \|\vec{g}\|_{L^p(\mathcal{X}, \ell^q)} < \infty.$$

(ii) The *homogeneous Hajlasz-Besov space* $\dot{N}_{p,q}^s(\mathcal{X})$ is the space of all measurable functions u such that

$$\|u\|_{\dot{N}_{p,q}^s(\mathcal{X})} \equiv \inf_{\vec{g} \in \mathbb{D}^s(u)} \|\vec{g}\|_{\ell^q(L^p(\mathcal{X}))} < \infty.$$

Some properties and useful characterizations of $\dot{M}_{p,q}^s(\mathcal{X})$ and $\dot{N}_{p,q}^s(\mathcal{X})$ are given in Section 2. In particular, denote by $\dot{M}^{s,p}(\mathcal{X})$ the Hajlasz-Sobolev space as in Definition 2.2. Then $\dot{M}^{s,p}(\mathcal{X}) = \dot{M}_{p,\infty}^s(\mathcal{X})$ for $s, p \in (0, \infty)$ as proved in Proposition 2.1.

Theorem 1.2. Let $n \in \mathbb{N}$.

- (i) If $s \in (0, 1)$, $p \in (n/(n+s), \infty)$ and $q \in (n/(n+s), \infty]$, then $\dot{M}_{p,q}^s(\mathbb{R}^n) = \dot{F}_{p,q}^s(\mathbb{R}^n)$.
- (ii) If $s \in (0, 1)$, $p \in (n/(n+s), \infty)$ and $q \in (0, \infty]$, then $\dot{N}_{p,q}^s(\mathbb{R}^n) = \dot{B}_{p,q}^s(\mathbb{R}^n)$.

The equivalences above are proven via grand Besov spaces $\mathcal{AB}_{p,q}^s(\mathbb{R}^n)$ and grand Triebel-Lizorkin spaces $\mathcal{AF}_{p,q}^s(\mathbb{R}^n)$ defined in Definition 3.2 below; see Section 3. Theorem 1.2 and Theorem 3.2 give pointwise characterizations for Besov and Triebel-Lizorkin spaces and have independent interest. For predecessors of such results, see [21, 31].

Relying on Theorem 2.1, Lemmas 2.1 and 2.3 below and several properties of quasiconformal mappings we obtain the following invariance property that, when combined with Theorem 1.2, yields Theorem 1.1; see Section 5.

Theorem 1.3. Let $n \geq 2$, $s \in (0, 1]$ and $q \in (0, \infty]$. Then $\dot{M}_{n/s,q}^s(\mathbb{R}^n)$ is invariant under quasiconformal mappings of \mathbb{R}^n .

The conclusion of Theorem 1.3 was previously only known in the case $s = 1$ and $q = \infty$; recall that $\dot{M}_{n,\infty}^1(\mathbb{R}^n) = \dot{M}^{1,n}(\mathbb{R}^n) = \dot{W}^{1,n}(\mathbb{R}^n)$.

Our results above also extend to a class of metric measure spaces. Indeed, let (\mathcal{X}, d, μ) be a metric measure space. For any $x \in \mathcal{X}$ and $r > 0$, let $B(x, r) \equiv \{y \in \mathcal{X} : d(x, y) < r\}$. Recall that (\mathcal{X}, d, μ) is called an *RD-space* in [14] if there exist constants $0 < C_1 \leq 1 \leq C_2$ and $0 < \kappa \leq n$ such that for all $x \in \mathcal{X}$, $0 < r < 2 \operatorname{diam} \mathcal{X}$ and $1 \leq \lambda < 2 \operatorname{diam} \mathcal{X}/r$,

$$(1.2) \quad C_1 \lambda^\kappa \mu(B(x, r)) \leq \mu(B(x, \lambda r)) \leq C_2 \lambda^n \mu(B(x, r)),$$

where and in what follows, $\operatorname{diam} \mathcal{X} \equiv \sup_{x, y \in \mathcal{X}} d(x, y)$; see [14]. In particular, if $\kappa = n$, then \mathcal{X} is called an *Ahlfors n -regular space*. Moreover, \mathcal{X} is said to support a *weak $(1, n)$ -Poincaré inequality* if there exists a positive constant C such that for all Lipschitz functions u ,

$$\int_B |u(x) - u_B| d\mu(x) \leq Cr \left\{ \int_{\lambda B} [\operatorname{Lip}(u(x))]^n d\mu(x) \right\}^{1/n}.$$

We then have a metric measure space version of Theorem 1.3 as follows.

Theorem 1.4. *Assume that \mathcal{X} and \mathcal{Y} are both Ahlfors n -regular spaces with $n > 1$, \mathcal{X} is proper and quasiconvex and supports a weak $(1, n)$ -Poincaré inequality and \mathcal{Y} is linearly locally connected. Let f be a quasiconformal mapping from \mathcal{X} onto \mathcal{Y} , which maps bounded sets into bounded sets. Then for every $s \in (0, 1]$, and for all $q \in (0, \infty]$, f induces an equivalence between $\dot{M}_{n/s, q}^s(\mathcal{X})$ and $\dot{M}_{n/s, q}^s(\mathcal{Y})$.*

The point is that, with the assumptions of Theorem 1.4, the quasiconformal mapping f is actually a quasisymmetric mapping and its volume derivative satisfies a suitable reverse Hölder inequality (see [16, Theorem 7.1], [17] and also Proposition 5.3 below), which allow us to extend the proof of Theorem 1.3 to this more general setting. In Theorem 1.4, both f and f^{-1} act as composition operators.

We also show, see Section 4, that the spaces $\dot{M}_{n/s, q}^s(\mathcal{X})$ and $\dot{M}_{n/s, q}^s(\mathcal{Y})$ identify with suitable Triebel-Lizorkin spaces and thus a version of the invariance of Triebel-Lizorkin spaces follows. Moreover, let us comment that our approach recovers the invariance of the Besov spaces considered by Bourdon and Pajot [2]; see Theorem 5.1 below.

Finally, we state some *conventions*. Throughout the paper, we denote by C a positive constant which is independent of the main parameters, but which may vary from line to line. Constants with subscripts, such as C_0 , do not change in different occurrences. The notation $A \lesssim B$ or $B \gtrsim A$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$. Denote by \mathbb{Z} the set of integers, \mathbb{N} the set of positive integers and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. For $\alpha \in \mathbb{R}$, denote by $[\alpha]$ the maximal integer no more than α . For any locally integrable function f , we denote by $\int_E f d\mu$ the average of f on E , namely, $\int_E f d\mu \equiv \frac{1}{\mu(E)} \int_E f d\mu$.

2 Some properties of $\dot{M}_{p, q}^s(\mathcal{X})$ and $\dot{N}_{p, q}^s(\mathcal{X})$

In this section, we establish some properties of $\dot{M}_{p, q}^s(\mathcal{X})$ and $\dot{N}_{p, q}^s(\mathcal{X})$, including the equivalence between $\dot{M}_{p, \infty}^s(\mathcal{X})$ and the Hajlasz-Sobolev space (see Proposition 2.1), several

equivalent characterizations for $\dot{M}_{p,q}^s(\mathcal{X})$ and $\dot{N}_{p,q}^s(\mathcal{X})$ (see Theorems 2.1 and 2.2), and Poincaré-type inequalities for $\dot{M}_{p,q}^s(\mathcal{X})$ with $\mathcal{X} = \mathbb{R}^n$ (see Lemmas 2.1 and 2.3).

First, we introduce the Hajłasz-Besov and Hajłasz-Triebel-Lizorkin spaces also in the case $p = \infty$ as follows.

Definition 2.1. Let (\mathcal{X}, d, μ) be a metric measure space, $s \in (0, \infty)$ and $q \in (0, \infty]$.

(i) The homogeneous Hajłasz-Triebel-Lizorkin space $\dot{M}_{\infty,q}^s(\mathcal{X})$ is the space of all measurable functions u such that $\|u\|_{\dot{M}_{\infty,q}^s(\mathcal{X})} < \infty$, where when $q < \infty$,

$$\|u\|_{\dot{M}_{\infty,q}^s(\mathcal{X})} \equiv \inf_{\vec{g} \in \mathbb{D}^s(u)} \sup_{k \in \mathbb{Z}} \sup_{x \in \mathcal{X}} \left\{ \sum_{j \geq k} \int_{B(x, 2^{-k})} [g_j(y)]^q d\mu(y) \right\}^{1/q}$$

and when $q = \infty$, $\|u\|_{\dot{M}_{\infty,\infty}^s(\mathcal{X})} \equiv \inf_{\vec{g} \in \mathbb{D}^s(u)} \|\vec{g}\|_{L^\infty(\mathcal{X}, \ell^\infty)}$.

(ii) The homogeneous Hajłasz-Besov space $\dot{N}_{\infty,q}^s(\mathcal{X})$ is the space of all measurable functions u such that

$$\|u\|_{\dot{N}_{\infty,q}^s(\mathcal{X})} \equiv \inf_{\vec{g} \in \mathbb{D}^s(u)} \|\vec{g}\|_{\ell^q(L^\infty(\mathcal{X}))} < \infty.$$

Then, we recall the definition of a Hajłasz-Sobolev space [11, 12] (see also [31] for a fractional version).

Let (\mathcal{X}, d, μ) be a metric measure space. For every $s \in (0, \infty)$ and measurable function u on \mathcal{X} , a non-negative function g is called an s -gradient of u if there exists a set $E \subset \mathcal{X}$ with $\mu(E) = 0$ such that for all $x, y \in \mathcal{X} \setminus E$,

$$(2.1) \quad |u(x) - u(y)| \leq [d(x, y)]^s [g(x) + g(y)].$$

Denote by $\mathcal{D}^s(u)$ the collection of all s -gradients of u .

Definition 2.2. Let $s \in (0, \infty)$ and $p \in (0, \infty]$. Then the homogeneous Hajłasz-Sobolev space $\dot{M}^{s,p}(\mathcal{X})$ is the set of all measurable functions u such that

$$\|u\|_{\dot{M}^{s,p}(\mathcal{X})} \equiv \inf_{g \in \mathcal{D}^s(u)} \|g\|_{L^p(\mathcal{X})} < \infty.$$

Hajłasz-Sobolev spaces naturally relate to Hajłasz-Triebel-Lizorkin spaces as follows.

Proposition 2.1. If $s \in (0, \infty)$ and $p \in (0, \infty]$, then $\dot{M}_{p,\infty}^s(\mathcal{X}) = \dot{M}^{s,p}(\mathcal{X})$.

Proof. Let $u \in \dot{M}^{s,p}(\mathcal{X})$ and $g \in \mathcal{D}^s(u)$. Taking $g_k \equiv g$, we know that $\vec{g} = \{g_k\}_{k \in \mathbb{Z}} \in \mathbb{D}^s(u)$ and $\|\vec{g}\|_{L^p(\mathcal{X}, \ell^q)} = \|g\|_{L^p(\mathcal{X})}$, which implies that $u \in \dot{M}_{p,\infty}^s(\mathcal{X})$ with $\|u\|_{\dot{M}_{p,\infty}^s(\mathcal{X})} = \|u\|_{\dot{M}^{s,p}(\mathcal{X})}$.

Conversely, let $u \in \dot{M}_{p,\infty}^s(\mathcal{X})$ and $\vec{g} \equiv \{g_k\}_{k \in \mathbb{Z}} \in \mathbb{D}^s(u)$. Taking $g \equiv \sup_{k \in \mathbb{Z}} g_k$, we have that $\|\vec{g}\|_{L^p(\mathcal{X}, \ell^q)} = \|g\|_{L^p(\mathcal{X})}$, which implies that $u \in \dot{M}^{s,p}(\mathcal{X})$ with $\|u\|_{\dot{M}^{s,p}(\mathcal{X})} = \|u\|_{\dot{M}_{p,\infty}^s(\mathcal{X})}$. This finishes the proof of Proposition 2.1. \square

Now, we introduce several useful variants of $\mathbb{D}^s(u)$ to characterize $\dot{M}_{p,q}^s(\mathcal{X})$ and $\dot{N}_{p,q}^s(\mathcal{X})$. To this end, let $s \in (0, \infty)$ and u be a measurable function on \mathcal{X} .

For $N_1, N_2 \in \mathbb{Z}_+$, denote by $\mathbb{D}^{s, N_1, N_2}(u)$ the *collection of all the sequences of nonnegative measurable functions*, $\vec{g} \equiv \{g_k\}_{k \in \mathbb{Z}}$, satisfying that there exists $E \subset \mathcal{X}$ with $\mu(E) = 0$ such that for all $k \in \mathbb{Z}$ and $x, y \in \mathcal{X} \setminus E$ with $2^{-k-1-N_1} \leq d(x, y) < 2^{-k+N_2}$,

$$|u(x) - u(y)| \leq [d(x, y)]^s [g_k(x) + g_k(y)].$$

For $\epsilon \in (0, s]$ and $N \in \mathbb{N}$, denote by $\tilde{\mathbb{D}}^{s, \epsilon, N}(u)$ the *collection of all the sequences of nonnegative measurable functions*, $\vec{g} \equiv \{g_k\}_{k \in \mathbb{Z}}$, satisfying that there exists $E \subset \mathcal{X}$ with $\mu(E) = 0$ such that for all $x, y \in \mathcal{X} \setminus E$,

$$|u(x) - u(y)| \leq [d(x, y)]^{s-\epsilon} \sum_{k \in \mathbb{Z}} 2^{-k\epsilon} [g_k(x) + g_k(y)] \chi_{[2^{-k-1-N}, \infty)}(d(x, y)).$$

For $\epsilon \in (0, \infty)$ and $N \in \mathbb{Z}$, denote by $\overline{\mathbb{D}}^{s, \epsilon, N}(u)$ the *collection of all the sequences of nonnegative measurable functions*, $\vec{g} \equiv \{g_k\}_{k \in \mathbb{Z}}$, satisfying that there exists $E \subset \mathcal{X}$ with $\mu(E) = 0$ such that for all $x, y \in \mathcal{X} \setminus E$,

$$|u(x) - u(y)| \leq [d(x, y)]^{s+\epsilon} \sum_{k \in \mathbb{Z}} 2^{k\epsilon} [g_k(x) + g_k(y)] \chi_{(0, 2^{-k-N})}(d(x, y)).$$

Then we have the following equivalent characterizations of $\dot{M}_{p,q}^s(\mathcal{X})$.

Theorem 2.1. (I) Let $s, p \in (0, \infty)$ and $q \in (0, \infty]$. Then the following are equivalent:

- (i) $u \in \dot{M}_{p,q}^s(\mathcal{X})$;
- (ii) for every pair of $N_1, N_2 \in \mathbb{Z}_+$, $\inf_{\vec{g} \in \mathbb{D}^{s, N_1, N_2}(u)} \|\vec{g}\|_{L^p(\mathcal{X}, \ell^q)} < \infty$;
- (iii) for every pair of $\epsilon_1 \in (0, s]$ and $N_3 \in \mathbb{Z}$, $\inf_{\vec{g} \in \tilde{\mathbb{D}}^{s, \epsilon_1, N_3}(u)} \|\vec{g}\|_{L^p(\mathcal{X}, \ell^q)} < \infty$;
- (iv) for every pair of $\epsilon_2 \in (0, \infty)$ and $N_4 \in \mathbb{Z}$, $\inf_{\vec{g} \in \overline{\mathbb{D}}^{s, \epsilon_2, N_4}(u)} \|\vec{g}\|_{L^p(\mathcal{X}, \ell^q)} < \infty$.

Moreover, given $\epsilon_1, \epsilon_2, N_1, N_2, N_3$ and N_4 as above, for all $u \in \dot{M}_{p,q}^s(\mathcal{X})$,

$$\begin{aligned} \|u\|_{\dot{M}_{p,q}^s(\mathcal{X})} &\sim \inf_{\vec{g} \in \mathbb{D}^{s, N_1, N_2}(u)} \|\vec{g}\|_{L^p(\mathcal{X}, \ell^q)} \\ &\sim \inf_{\vec{g} \in \tilde{\mathbb{D}}^{s, \epsilon_1, N_3}(u)} \|\vec{g}\|_{L^p(\mathcal{X}, \ell^q)} \sim \inf_{\vec{g} \in \overline{\mathbb{D}}^{s, \epsilon_2, N_4}(u)} \|\vec{g}\|_{L^p(\mathcal{X}, \ell^q)}, \end{aligned}$$

where the implicit constants are independent of u .

(II) Let $s \in (0, \infty)$ and $p, q \in (0, \infty]$. Then the above statements still hold with $\dot{M}_{p,q}^s(\mathcal{X})$ and $L^p(\mathcal{X}, \ell^q)$ replaced by $\dot{N}_{p,q}^s(\mathcal{X})$ and $\ell^q(L^p(\mathcal{X}))$, respectively.

Proof. We first prove that (i) implies (ii), (iii) and (iv). Let u be a measurable function and $\vec{g} \in \mathbb{D}^s(u)$. Then for every pair of $N_1, N_2 \in \mathbb{Z}_+$, setting $h_k \equiv \sum_{j=-N_2}^{N_1} g_{k+j}$ for $k \in \mathbb{Z}$, we know that $\vec{h} \equiv \{h_k\}_{k \in \mathbb{Z}} \in \mathbb{D}^{s, N_1, N_2}(u)$. For every pair of $\epsilon_1 \in (0, s]$ and $N_3 \in \mathbb{Z}$, taking $h_k \equiv 2^{N_3\epsilon_1} g_{k-N_3}$ for all $k \in \mathbb{Z}$, we have $\vec{h} \equiv \{h_k\}_{k \in \mathbb{Z}} \in \tilde{\mathbb{D}}^{s, \epsilon_1, N_3}(u)$. For every pair of $\epsilon_2 \in (0, \infty)$ and $N_4 \in \mathbb{Z}$, taking $h_k \equiv 2^{N_4\epsilon_2} g_{k-N_4}$ for all $k \in \mathbb{Z}$, we have $\vec{h} \equiv \{h_k\}_{k \in \mathbb{Z}} \in \overline{\mathbb{D}}^{s, \epsilon_2, N_4}(u)$. Then it is to easy to see that in all of the above cases, we

have $\|\vec{h}\|_{L^p(\mathcal{X}, \ell^q)} \lesssim \|u\|_{\dot{M}_{p,q}^s(\mathcal{X})}$ and $\|\vec{h}\|_{\ell^q(L^p(\mathcal{X}))} \lesssim \|u\|_{\dot{N}_{p,q}^s(\mathcal{X})}$. Thus, (i) implies (ii), (iii) and (iv).

Now we prove the converse. Since $\mathbb{D}^{s, N_1, N_2}(u) \subset \mathbb{D}^s(u)$, we have that (ii) implies (i).

To show that (iii) implies (i), let u be a measurable function and $\vec{g} \in \widetilde{\mathbb{D}}^{s, \epsilon_1, N_3}(u)$. For all $k \in \mathbb{Z}$, set $h_k \equiv \sum_{j=k-N_3}^{\infty} 2^{(k-j+1)\epsilon_1} g_j$. Then $\vec{h} \equiv \{h_k\}_{k \in \mathbb{Z}} \in \mathbb{D}^s(u)$.

Moreover, if $p \in (0, \infty)$, by the Hölder inequality when $q \in (1, \infty)$ and the inequality

$$(2.2) \quad \left(\sum_{i \in \mathbb{Z}} |a_i| \right)^q \leq \sum_i |a_i|^q$$

for $\{a_i\}_{i \in \mathbb{Z}} \subset \mathbb{R}$ when $q \in (0, 1]$, we have

$$\|\vec{h}\|_{\ell^q}^q \lesssim \sum_{k \in \mathbb{Z}} \left[\sum_{j=k-N_3}^{\infty} 2^{(k-j+1)\epsilon_1} g_j \right]^q \lesssim \sum_{k \in \mathbb{Z}} \sum_{j=k-N_3}^{\infty} 2^{-(j-k)q\epsilon_1/2} [g_j]^q \lesssim \|\vec{g}\|_{\ell^q}^q,$$

which gives that $\|\vec{h}\|_{L^p(\mathcal{X}, \ell^q)} \lesssim \|u\|_{\dot{M}_{p,q}^s(\mathcal{X})}$. This also holds when $q = \infty$, as seen with a slight modification.

On the other hand, by the Hölder inequality when $p \in (1, \infty)$ and the inequality (2.2) with $q = p$ when $p \in (0, 1]$, we have

$$\begin{aligned} \|\vec{h}\|_{\ell^q(L^p(\mathcal{X}))}^q &\lesssim \sum_{k \in \mathbb{Z}} \left(\int_{\mathcal{X}} \left[\sum_{j=k-N_3}^{\infty} 2^{-(j-k-1)\epsilon_1} g_j(y) \right]^p d\mu(y) \right)^{q/p} \\ &\lesssim \sum_{k \in \mathbb{Z}} \left(\sum_{j=k-N_3}^{\infty} 2^{-(j-k)p\epsilon_1/2} \int_{\mathcal{X}} [g_j(y)]^p d\mu(y) \right)^{q/p}. \end{aligned}$$

Applying the Hölder inequality when $p/q \in (1, \infty)$ and the inequality (2.2) with power q/p instead of q again when $p/q \in (0, 1]$, we further have

$$\|\vec{h}\|_{\ell^q(L^p(\mathcal{X}))}^q \lesssim \sum_{k \in \mathbb{Z}} \sum_{j=k-N_3}^{\infty} 2^{-(j-k)q\epsilon_1/4} \left(\int_{\mathcal{X}} [g_j(y)]^p d\mu(y) \right)^{q/p} \lesssim \|\vec{g}\|_{\ell^q(L^p(\mathcal{X}))}^q,$$

which gives that $\|\vec{h}\|_{\ell^q(L^p(\mathcal{X}))} \lesssim \|u\|_{\dot{N}_{p,q}^s(\mathcal{X})}$. This also holds when $p = \infty$ or $q = \infty$, as easily seen.

To prove that (iv) implies (i), let u be a measurable function and $\vec{g} \in \widetilde{\mathbb{D}}^{s, \epsilon_2, N_4}(u)$. For every $k \in \mathbb{Z}$, set $h_k \equiv \sum_{j=-\infty}^{k-N_4} 2^{(j-k)\epsilon_2} g_j$. Then $\vec{h} \equiv \{h_k\}_{k \in \mathbb{Z}} \in \mathbb{D}^s(u)$.

Moreover, if $p \in (0, \infty)$, by the Hölder inequality when $q \in (1, \infty)$ and the inequality (2.2) when $q \in (0, 1]$, we have

$$\|\vec{h}\|_{\ell^q}^q \lesssim \sum_{k \in \mathbb{Z}} \left[\sum_{j=-\infty}^{k-N_4} 2^{(j-k)\epsilon_2} g_j \right]^q \lesssim \sum_k \sum_{j=-\infty}^{k-N_4} 2^{(j-k)q\epsilon_2/2} [g_j]^q \lesssim \|\vec{g}\|_{\ell^q}^q,$$

which gives that $\|\vec{h}\|_{L^p(\mathcal{X}, \ell^q)} \lesssim \|u\|_{\dot{M}_{p,q}^s(\mathcal{X})}$. This also extends to the case $q = \infty$.

Similarly, one can prove that $\|\vec{h}\|_{\ell^q(L^p(\mathcal{X}))} \lesssim \|u\|_{\dot{N}_{p,q}^s(\mathcal{X})}$, but we omit the details. This finishes the proof of Theorem 2.1. \square

Theorem 2.2. *Let $s \in (0, 1]$ and $q \in (0, \infty)$. Then the following are equivalent:*

- (i) $u \in \dot{M}_{\infty,q}^s(\mathcal{X})$;
- (ii) for every $N_2 \in \mathbb{Z}_+$,

$$(2.3) \quad \inf_{\vec{g} \in \mathbb{D}^{s,0,N_2}(u)} \sup_{k \in \mathbb{Z}} \sup_{x \in \mathcal{X}} \left(\sum_{j \geq k} \int_{B(x, 2^{-k})} [g_j(y)]^q d\mu(y) \right)^{1/q} < \infty;$$

- (iii) for every pair of $\epsilon \in (0, s]$ and $N_3 \in \mathbb{Z} \setminus \mathbb{N}$,

$$(2.4) \quad \inf_{\vec{g} \in \widetilde{\mathbb{D}}^{s,\epsilon,N_3}(u)} \sup_{k \in \mathbb{Z}} \sup_{x \in \mathcal{X}} \left(\sum_{j \geq k} \int_{B(x, 2^{-k})} [g_j(y)]^q d\mu(y) \right)^{1/q} < \infty.$$

Moreover, given ϵ , N_2 and N_3 as above, for all $u \in \dot{M}_{\infty,q}^s(\mathcal{X})$, $\|u\|_{\dot{M}_{\infty,q}^s(\mathcal{X})}$ is equivalent to the given quantity.

Proof. We first prove that (i) implies (ii) and (iii). Let u be a measurable function and $\vec{g} \in \mathbb{D}^s(u)$. Then for every $N_2 \in \mathbb{Z}_+$, setting $h_k \equiv \sum_{j=-N_2}^0 g_{k+j}$ for all $k \in \mathbb{Z}$, we know that $\vec{h} \equiv \{h_k\}_{k \in \mathbb{Z}} \in \mathbb{D}^{s,0,N_2}(u)$. For every pair of $\epsilon \in (0, s]$ and $N_3 \in \mathbb{Z} \setminus \mathbb{N}$, taking $h_k \equiv 2^{N_3\epsilon} g_{k-N_3}$ for all $k \in \mathbb{Z}$, we have $\vec{h} \equiv \{h_k\}_{k \in \mathbb{Z}} \in \widetilde{\mathbb{D}}^{s,\epsilon,N_3}(u)$. Then it is easy to see that in both cases,

$$\sum_{j \geq k} \int_{B(x, 2^{-k})} [h_j(y)]^q d\mu(y) \lesssim \sum_{j \geq k} \int_{B(x, 2^{-k})} [g_j(y)]^q d\mu(y).$$

Thus, (i) implies (ii) and (iii).

Conversely, since $\mathbb{D}^{s,0,N_2}(u) \subset \mathbb{D}^s(u)$, we have that (ii) implies (i).

To prove that (iii) implies (i), let u be a measurable function and $\vec{g} \in \widetilde{\mathbb{D}}^{s,\epsilon,N_3}(u)$. For all $k \in \mathbb{Z}$, set $h_k \equiv \sum_{j=k-N_3}^{\infty} 2^{(k-j+1)\epsilon} g_j$. Then $\vec{h} \equiv \{h_k\}_{k \in \mathbb{Z}} \in \mathbb{D}^s(u)$. For all $x \in \mathcal{X}$ and $k \in \mathbb{Z}$, by the Hölder inequality when $q \in (1, \infty)$ and the inequality (2.2) when $q \in (0, 1]$, we have

$$\sum_{j \geq k} [h_j]^q \sim \sum_{j \geq k} \left[\sum_{i=j-N_3}^{\infty} 2^{-(i-j)\epsilon} g_i \right]^q \lesssim \sum_{j \geq k} \sum_{i=j-N_3}^{\infty} 2^{-(i-j)q\epsilon/2} [g_i]^q \lesssim \sum_{i \geq k-N_3} [g_i]^q,$$

which together with $N_3 \leq 0$ implies that

$$\sum_{j \geq k} \int_{B(x, 2^{-k})} [h_j(y)]^q d\mu(y) \lesssim \sum_{i \geq k} \int_{B(x, 2^{-k})} [g_i(y)]^q d\mu(y).$$

Thus, (iii) implies (i). This finishes the proof of Theorem 2.2. \square

Remark 2.1. Comparing to Theorem 2.1, notice that we require $N_1 = 0$ and $N_3 \leq 0$ in Theorem 2.2. However, if \mathcal{X} has the doubling property, then Theorem 2.2 still holds for all $N_1, N_2 \in \mathbb{Z}_+$ and $N_3 \in \mathbb{Z}$. We omit the details.

Finally, let (\mathcal{X}, d, μ) be \mathbb{R}^n endowed with the Lebesgue measure and the Euclidean distance. The following Poincaré-type inequalities for $\dot{M}_{p,q}^s(\mathbb{R}^n)$ play an important role in the following.

Lemma 2.1. *Let $s \in (0, 1]$, $p \in (1, \infty]$ and $q \in (0, \infty]$. Then there exists a positive constant C such that for all $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$, $u \in \dot{M}_{p,q}^s(B(x, 2^{-k+2}))$ and $\vec{g} \in \mathbb{D}^s(u)$,*

$$\inf_{c \in \mathbb{R}} \int_{B(x, 2^{-k})} |u(y) - c| dy \leq C 2^{-ks} \sum_{j=k-3}^k \int_{B(x, 2^{-k+2})} g_j(y) dy.$$

Proof. Notice that for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$,

$$\begin{aligned} \inf_{c \in \mathbb{R}} \int_{B(x, 2^{-k})} |u(y) - c| dy &\leq \int_{B(x, 2^{-k})} |u(y) - u_{B(x, 2^{-k+2}) \setminus B(x, 2^{-k+1})}| dy \\ &\leq \int_{B(x, 2^{-k})} \int_{B(x, 2^{-k+2}) \setminus B(x, 2^{-k+1})} |u(y) - u(z)| dy dz. \end{aligned}$$

Since for $y \in B(x, 2^{-k})$ and $z \in B(x, 2^{-k+2}) \setminus B(x, 2^{-k+1})$, we have that $2^{-k} \leq |y - z| < 2^{-k+3}$, which implies that

$$|u(y) - u(z)| \lesssim 2^{-ks} \sum_{j=k-3}^{k-1} [g_j(y) + g_j(z)].$$

Thus,

$$\inf_{c \in \mathbb{R}} \int_{B(x, 2^{-k})} |u(y) - c| dy \lesssim 2^{-ks} \sum_{j=k-3}^{k-1} \int_{B(x, 2^{-k+2})} g_j(y) dy,$$

which completes the proof of Lemma 2.1. \square

The following inequality was given by Hajlasz [12, Theorem 8.7] when $s = 1$, and when $s \in (0, 1)$, it can be proved by a slight modification of the proof of [12, Theorem 8.7].

Lemma 2.2. *Let $s \in (0, 1]$, $p \in (0, n/s)$ and $p_* = np/(n-sp)$. Then there exists a positive constant C such that for all $x \in \mathbb{R}^n$, $r \in (0, \infty)$, $u \in \dot{M}^{s,p}(B(x, 2r))$ and $g \in \mathcal{D}^s(u)$,*

$$\inf_{c \in \mathbb{R}} \left(\int_{B(x, r)} |u(y) - c|^{p_*} dy \right)^{1/p_*} \leq C r^s \left(\int_{B(x, 2r)} [g(y)]^p dy \right)^{1/p}.$$

Lemma 2.3. *Let $s \in (0, 1]$ and $p \in (0, 1]$. Then for every pair of $\epsilon, \epsilon' \in (0, s)$ with $\epsilon < \epsilon'$, there exists a positive constant C such that for all $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$, measurable functions u and $\vec{g} \in \mathbb{D}^s(u)$,*

$$(2.5) \quad \inf_{c \in \mathbb{R}} \left(\int_{B(x, 2^{-k})} |u(y) - c|^{np/(n-\epsilon p)} dy \right)^{(n-\epsilon p)/(np)} \\ \leq C 2^{-k\epsilon'} \sum_{j \geq k-2} 2^{-j(s-\epsilon')} \left\{ \int_{B(x, 2^{-k+1})} [g_j(y)]^p dy \right\}^{1/p}.$$

Proof. For given $\epsilon, \epsilon' \in (0, s)$ with $\epsilon < \epsilon'$, and all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, without loss of generality, we may assume that the right-hand side of (2.5) is finite. For $\vec{g} \in \mathbb{D}^s(u)$, taking $g \equiv \left\{ \sum_{j \geq k-2} 2^{-j(s-\epsilon)p} (g_j)^p \right\}^{1/p}$, we have that $g \in \mathcal{D}^\epsilon(u)$ and $u \in \dot{M}^{\epsilon, p}(B(x, 2^{-k+1}))$. Indeed, for every pair of $y, z \in B(x, 2^{-k+1})$, there exists $j \geq k-2$ such that $2^{-j-1} \leq |y - z| < 2^{-j}$ and hence

$$|u(y) - u(z)| \leq |y - z|^s [g_j(y) + g_j(z)] \leq |y - z|^\epsilon [g(y) + g(z)].$$

Moreover, by (2.2) with $q = p$, $\epsilon < \epsilon'$ and the Hölder inequality, we have

$$(2.6) \quad \|g\|_{L^p(B(x, 2^{-k+1}))} \\ \leq \left\{ \sum_{j \geq k-2} 2^{-j(s-\epsilon)p} \int_{B(x, 2^{-k+1})} [g_j(y)]^p dy \right\}^{1/p} \\ \lesssim \left(\sum_{j \geq k-2} 2^{-j(\epsilon' - \epsilon)p/(1-p)} \right)^{(1-p)/p} \sum_{j \geq k-2} 2^{-j(s-\epsilon')} \left\{ \int_{B(x, 2^{-k+1})} [g_j(y)]^p dy \right\}^{1/p} \\ \lesssim 2^{-k(\epsilon' - \epsilon + n/p)} \sum_{j \geq k-2} 2^{-j(s-\epsilon')} \left\{ \int_{B(x, 2^{-k+1})} [g_j(y)]^p dy \right\}^{1/p}.$$

Thus, the above claims are true.

Then, applying Lemma 2.2, we obtain

$$\inf_{c \in \mathbb{R}} \left(\int_{B(x, 2^{-k})} |u(y) - c|^{np/(n-\epsilon p)} dy \right)^{(n-\epsilon p)/(np)} \\ \lesssim 2^{-k\epsilon} \left(\int_{B(x, 2^{-k+1})} [g(y)]^p dy \right)^{1/p} \\ \lesssim 2^{-k\epsilon'} \sum_{j \geq k-2} 2^{-j(s-\epsilon')} \left\{ \int_{B(x, 2^{-k+1})} [g_j(y)]^p dy \right\}^{1/p},$$

which together with (2.6) gives (2.5). This finishes the proof of Lemma 2.3. \square

Remark 2.2. From Lemmas 2.1 through 2.3, it is easy to see that for all $s \in (0, 1]$, $p \in (n/(n+s), \infty]$ and $q \in (0, \infty]$, the elements of $\dot{M}_{p,q}^s(\mathbb{R}^n)$ are actually locally integrable.

3 Besov and Triebel-Lizorkin spaces on \mathbb{R}^n

In this section, with the aid of grand Littlewood-Paley functions, we characterize full ranges of the classical Besov and Triebel-Lizorkin spaces on \mathbb{R}^n with $n \in \mathbb{N}$ and establish their equivalence with the Hajlasz-Besov and Hajlasz-Triebel-Lizorkin spaces; see Theorems 3.1 and 3.2. In particular, Theorem 1.2 follows from (i) and (ii) of Theorem 3.2 with $p \in (0, \infty)$.

We first recall some notions and notation. In this section, we work on \mathbb{R}^n with $n \in \mathbb{N}$. Recall that $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions, whose topology is determined by a family of seminorms, $\{\|\cdot\|_{\mathcal{S}_{k,m}(\mathbb{R}^n)}\}_{k,m \in \mathbb{Z}_+}$, where for all $k \in \mathbb{Z}_+$, $m \in (0, \infty)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\|\varphi\|_{\mathcal{S}_{k,m}(\mathbb{R}^n)} \equiv \sup_{\alpha \in \mathbb{Z}_+^n, |\alpha| \leq k} \sup_{x \in \mathbb{R}^n} (1 + |x|)^m |\partial^\alpha \varphi(x)|.$$

Here, for any $\alpha \equiv (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\partial^\alpha \equiv (\frac{\partial}{\partial x_1})^{\alpha_1} \dots (\frac{\partial}{\partial x_n})^{\alpha_n}$. It is known that $\mathcal{S}(\mathbb{R}^n)$ forms a locally convex topological vector space. Denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of $\mathcal{S}(\mathbb{R}^n)$ endowed with the weak $*$ -topology. In what follows, for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $t > 0$ and $x \in \mathbb{R}^n$, set $\varphi_t(x) \equiv t^{-n} \varphi(t^{-1}x)$.

Then the classical Besov and Triebel-Lizorkin spaces are defined as follows; see [27].

Definition 3.1. Let $s \in \mathbb{R}$ and $p, q \in (0, \infty]$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy that

$$(3.1) \quad \text{supp } \widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\} \text{ and } |\widehat{\varphi}(\xi)| \geq \text{constant} > 0 \text{ if } 3/5 \leq |\xi| \leq 5/3.$$

(i) The homogeneous Triebel-Lizorkin space $\dot{F}_{p,q}^s(\mathbb{R}^n)$ is defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{\dot{F}_{p,q}^s(\mathbb{R}^n)} < \infty$, where when $p < \infty$,

$$\|f\|_{\dot{F}_{p,q}^s(\mathbb{R}^n)} \equiv \left\| \left(\sum_{k \in \mathbb{Z}} 2^{ksq} |\varphi_{2^{-k}} * f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

with the usual modification made when $q = \infty$, and when $p = \infty$,

$$\|f\|_{\dot{F}_{\infty,q}^s(\mathbb{R}^n)} \equiv \sup_{x \in \mathbb{R}^n} \sup_{\ell \in \mathbb{Z}} \left(\int_{B(x, 2^{-\ell})} \sum_{k \geq \ell} 2^{ksq} |\varphi_{2^{-k}} * f(y)|^q d\mu(y) \right)^{1/q}$$

with the usual modification made when $q = \infty$.

(ii) The homogeneous Besov space $\dot{B}_{p,q}^s(\mathbb{R}^n)$ is defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} \equiv \left(\sum_{k \in \mathbb{Z}} 2^{ksq} \|\varphi_{2^{-k}} * f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q}$$

with the usual modification made when $q = \infty$.

Remark 3.1. Notice that if $\|f\|_{\dot{F}_{p,q}^s(\mathbb{R}^n)} = 0$, then it is easy to see that f is a polynomial. Denote by \mathcal{P} the collection of all polynomials on \mathbb{R}^n . So the quotient space $\dot{F}_{p,q}^s(\mathbb{R}^n)/\mathcal{P}$ is a quasi-Banach space. By abuse of the notation, the space $\dot{F}_{p,q}^s(\mathbb{R}^n)/\mathcal{P}$ is always denoted by $\dot{F}_{p,q}^s(\mathbb{R}^n)$, and its element $[f] = f + \mathcal{P}$ with $f \in \dot{F}_{p,q}^s(\mathbb{R}^n)$ simply by f . A similar observation is also suitable to $\dot{B}_{p,q}^s(\mathbb{R}^n)$.

Moreover, for each $N \in \mathbb{Z}_+$, denote by $\mathcal{S}_N(\mathbb{R}^n)$ the space of all functions $f \in \mathcal{S}(\mathbb{R}^n)$ satisfying that $\int_{\mathbb{R}^n} x^\alpha f(x) dx = 0$ for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq N$. For convenience, we also write $\mathcal{S}_{-1}(\mathbb{R}^n) \equiv \mathcal{S}(\mathbb{R}^n)$.

For each $N \in \mathbb{Z}_+ \cup \{-1\}$, $m \in (0, \infty)$ and $\ell \in \mathbb{Z}_+$, we define the class $\mathcal{A}_{N,m}^\ell$ of test functions by

$$(3.2) \quad \mathcal{A}_{N,m}^\ell \equiv \{\phi \in \mathcal{S}_N(\mathbb{R}^n) : \|\phi\|_{\mathcal{S}_{N+\ell+1,m}(\mathbb{R}^n)} \leq 1\}.$$

Then the grand Besov and Triebel-Lizorkin spaces are defined as follows.

Definition 3.2. Let $s \in \mathbb{R}$ and $q \in (0, \infty]$. Let \mathcal{A} be a class of test functions as in (3.2).

(i) The homogeneous grand Triebel-Lizorkin space $\mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n)$ is defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{\mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n)} < \infty$, where $\|f\|_{\mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n)}$ is defined as in $\|f\|_{\dot{F}_{p,q}^s(\mathbb{R}^n)}$ via replacing $|\varphi_{2^{-k}} * u|$ by $\sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u|$.

(ii) The homogeneous grand Besov space $\mathcal{A}\dot{B}_{p,q}^s(\mathbb{R}^n)$ is defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{\mathcal{A}\dot{B}_{p,q}^s(\mathbb{R}^n)} < \infty$, where $\|f\|_{\mathcal{A}\dot{B}_{p,q}^s(\mathbb{R}^n)}$ is defined as in $\|f\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)}$ via replacing $|\varphi_{2^{-k}} * u|$ by $\sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u|$.

Remark 3.2. For $\mathcal{A} \equiv \mathcal{A}_{N,m}^\ell$, we also write $\mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n)$ as $\mathcal{A}_{N,m}^\ell \dot{F}_{p,q}^s(\mathbb{R}^n)$. Moreover, if $N \in \mathbb{Z}_+$ and $\|f\|_{\mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n)} = 0$, then it is easy to see that $f \in \mathcal{P}_N$, where \mathcal{P}_N is the space of polynomials with degree no more than N . So, similarly to Remark 3.1, the quotient space $\mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n)/\mathcal{P}_N$ is always denoted by $\mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n)$ and its element $[f] = f + \mathcal{P}$ with $f \in \mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n)$ simply by f . A similar observation is also suitable to $\mathcal{A}\dot{B}_{p,q}^s(\mathbb{R}^n)$.

The main results of this section read as follows.

Theorem 3.1. Let $s \in \mathbb{R}$ and $p, q \in (0, \infty]$.

(i) If $J \equiv n/\min\{1, p, q\}$, $\mathcal{A} = \mathcal{A}_{N,m}^\ell$ with $\ell \in \mathbb{Z}_+$, $N+1 > \max\{s, J-n-s\}$ and $m > \max\{J, n+N+1\}$, then $\mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n) = \dot{F}_{p,q}^s(\mathbb{R}^n)$.

(ii) If $J \equiv n/\min\{1, p\}$, $\mathcal{A} = \mathcal{A}_{N,m}^\ell$ with $\ell \in \mathbb{Z}_+$, $N+1 > \max\{s, J-n-s\}$ and $m > \max\{J, n+N+1\}$, then $\mathcal{A}\dot{B}_{p,q}^s(\mathbb{R}^n) = \dot{B}_{p,q}^s(\mathbb{R}^n)$.

Theorem 3.2. Let $\mathcal{A} \equiv \mathcal{A}_{0,m}^\ell$ with $\ell \in \mathbb{Z}_+$ and $m > n+1$.

(i) If $s \in (0, 1)$ and $p, q \in (n/(n+s), \infty]$, then $\dot{M}_{p,q}^s(\mathbb{R}^n) = \dot{F}_{p,q}^s(\mathbb{R}^n)$.

(ii) If $s \in (0, 1)$, $p \in (n/(n+s), \infty]$ and $q \in (0, \infty]$, then $\dot{N}_{p,q}^s(\mathbb{R}^n) = \dot{B}_{p,q}^s(\mathbb{R}^n)$.

(iii) If $s \in (0, 1]$ and $p, q \in (n/(n+s), \infty]$, then $\dot{M}_{p,q}^s(\mathbb{R}^n) = \mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n)$.

(iv) If $s \in (0, 1]$, $p \in (n/(n+s), \infty]$ and $q \in (0, \infty]$, then $\dot{N}_{p,q}^s(\mathbb{R}^n) = \mathcal{A}\dot{B}_{p,q}^s(\mathbb{R}^n)$.

Remark 3.3. (i) Recall that Theorem 3.1 for $\dot{F}_{p,q}^s(\mathbb{R}^n)$ with $p < \infty$ was already given in [22, Theorem 1.2]. The proof of Theorem 3.1 for the full range of Besov and Triebel-Lizorkin spaces is similar to that of [22, Theorem 1.2]. For the reader's convenience, we sketch it below.

(ii) For all $s \in (0, 1)$ and $p \in (n/(n+s), \infty)$, combining [31, Corollary 1.3], [22, Corollary 1.2] and Proposition 2.1, we already have $\dot{M}_{p,\infty}^s(\mathbb{R}^n) = \dot{M}^{s,p}(\mathbb{R}^n) = \dot{F}_{p,\infty}^s(\mathbb{R}^n)$.

(iii) When $s = 1$, as proved in [11, 21], $\dot{M}^{1,p}(\mathbb{R}^n) = \dot{W}^{1,p}(\mathbb{R}^n)$ for $p \in (1, \infty)$ and $\dot{M}^{1,p}(\mathbb{R}^n) = \dot{H}^{1,p}(\mathbb{R}^n)$ for $p \in (n/(n+1), 1]$, which together with Proposition 2.1 and [27] implies that $\dot{M}_{p,\infty}^1(\mathbb{R}^n) = \dot{M}^{1,p}(\mathbb{R}^n) = \dot{F}_{p,2}^1(\mathbb{R}^n)$ for all $p \in (n/(n+1), \infty)$. Here $\dot{W}^{1,p}(\mathbb{R}^n)$ with $p \in (1, \infty)$ denotes the homogeneous Sobolev space and $\dot{H}^{1,p}(\mathbb{R}^n)$ with $p \in (0, 1]$ the homogeneous Hardy-Sobolev space.

Proof of Theorem 3.1. Notice that $\|u\|_{\dot{F}_{p,q}^s(\mathbb{R}^n)} \lesssim \|u\|_{\mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n)}$ for all $u \in \mathcal{S}'(\mathbb{R}^n)$, which implies that $\mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n) \subset \dot{F}_{p,q}^s(\mathbb{R}^n)$. Similarly, $\mathcal{A}\dot{B}_{p,q}^s(\mathbb{R}^n) \subset \dot{B}_{p,q}^s(\mathbb{R}^n)$. Conversely, assume that $u \in \dot{F}_{p,q}^s(\mathbb{R}^n)$ or $u \in \dot{B}_{p,q}^s(\mathbb{R}^n)$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy the same conditions as φ and $\sum_{k \in \mathbb{Z}} \widehat{\varphi}(2^{-k}\xi) \widehat{\psi}(2^{-k}\xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$; see [7, Lemma (6.9)] for the existence of ψ . Then, by the Calderón reproducing formula, for $f \in \mathcal{S}'(\mathbb{R}^n)$, there exist polynomials P_u and $\{P_i\}_{i \in \mathbb{Z}}$ depending on f such that

$$(3.3) \quad u + P_u = \lim_{i \rightarrow -\infty} \left\{ \sum_{j=i}^{\infty} \varphi_{2^{-j}} * \psi_{2^{-j}} * u + P_i \right\},$$

where the series converges in $\mathcal{S}'(\mathbb{R}^n)$; see, for example, [24, 5].

Moreover, if $u \in \dot{F}_{p,q}^s(\mathbb{R}^n)$ with $p \in (0, \infty)$, then it is known that the degrees of the polynomials $\{P_i\}_{i \in \mathbb{Z}}$ here are no more than $\lfloor s - n/p \rfloor$; see [6, pp.153-155] and [5]. Furthermore, as shown in [6, pp.153-155], $u + P_u$ is the canonical representative of u in the sense that if $i = 1, 2$, $\varphi^{(i)}, \psi^{(i)}$ satisfy (3.1) and $\sum_{k \in \mathbb{Z}} \widehat{\varphi}^{(i)}(2^{-k}\xi) \widehat{\psi}^{(i)}(2^{-k}\xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, then $P_u^{(1)} - P_u^{(2)}$ is a polynomial of degree no more than $\lfloor s - n/p \rfloor$, where $P_u^{(i)}$ is as in (3.3) corresponding to $\varphi^{(i)}, \psi^{(i)}$ for $i = 1, 2$. So, in this sense, we identify u with $\tilde{u} \equiv u + P_u$.

We point out that the above argument still holds when $u \in \dot{B}_{p,q}^s(\mathbb{R}^n)$ or $u \in \dot{F}_{p,q}^s(\mathbb{R}^n)$ with the full range. In fact, by [24, pp.52-56], if $u \in \dot{B}_{p,\infty}^s(\mathbb{R}^n)$, then the above arguments hold. Moreover, by $\dot{F}_{\infty,q}^s(\mathbb{R}^n) \subset \dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$ and $\dot{B}_{p,q}^s(\mathbb{R}^n) \subset \dot{B}_{p,\infty}^s(\mathbb{R}^n)$ for all possible s, p and q , the above arguments hold for all $u \in \dot{B}_{p,q}^s(\mathbb{R}^n)$ or $u \in \dot{F}_{p,q}^s(\mathbb{R}^n)$ with the full range.

Let $\tilde{\varphi}(x) \equiv \varphi(-x)$ for all $x \in \mathbb{R}^n$. Denote by \mathcal{Q} the collection of all dyadic cubes on \mathbb{R}^n . For every dyadic cube $Q \equiv 2^{-j}k + 2^{-j}[0, 1]^n$ with certain $k \in \mathbb{Z}^n$, set $x_Q \equiv 2^{-j}k$, denote by $\ell(Q) \equiv 2^{-j}$ the side length of Q and write $\varphi_Q(x) \equiv 2^{jn/2} \varphi(2^j x - k) = 2^{-jn/2} \varphi_{2^{-j}}(x - x_Q)$ for all $x \in \mathbb{R}^n$. Then for all $u \in \dot{F}_{p,q}^s(\mathbb{R}^n)$ or $u \in \dot{B}_{p,q}^s(\mathbb{R}^n)$, $\phi \in \mathcal{S}_N(\mathbb{R}^n)$ with $N \geq \lfloor s - n/p \rfloor$, $i \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, by [5, 7], [3, Lemma 2.8] and an argument as in the proof of [22, Theorem 1.2], we have

$$\tilde{u} * \phi_{2^{-i}}(x) = \sum_{Q \in \mathcal{Q}} \langle u, \tilde{\varphi}_Q \rangle \psi_Q * \phi_{2^{-i}}(x) = \sum_{Q \in \mathcal{Q}} t_Q \psi_Q * \phi_{2^{-i}}(x),$$

where $t_Q = \langle u, \tilde{\varphi}_Q \rangle$. Moreover, by the proof of [22, Theorem 1.2] again, for all $R \in \mathcal{Q}$ with $\ell(R) = 2^{-i}$, we have

$$|\tilde{u} * \phi_{2^{-i}}| \lesssim \sum_{\ell(R)=2^{-i}} \left(\sum_{Q \in \mathcal{Q}} a_{RQ} t_Q \right) |R|^{-1/2} \chi_R,$$

where

$$a_{RQ} \leq \left[\frac{\ell(R)}{\ell(Q)} \right]^s \left[1 + \frac{|x_R - x_Q|}{\max\{\ell(R), \ell(Q)\}} \right]^{-J-\epsilon} \min \left\{ \left[\frac{\ell(R)}{\ell(Q)} \right]^{\frac{n+\epsilon}{2}}, \left[\frac{\ell(Q)}{\ell(R)} \right]^{J+\frac{\epsilon-n}{2}} \right\}$$

for certain $\epsilon > 0$. If $J \equiv n/\min\{1, p, q\}$, then $\{a_{RQ}\}_{R, Q \in \mathcal{Q}}$ forms an almost diagonal operator on $\dot{f}_{p,q}^s(\mathbb{R}^n)$ and hence, is bounded on $\dot{f}_{p,q}^s(\mathbb{R}^n)$, while if $J \equiv n/\min\{1, p\}$, then $\{a_{RQ}\}_{R, Q \in \mathcal{Q}}$ forms an almost diagonal operator on $\dot{f}_{p,q}^s(\mathbb{R}^n)$ and hence, is bounded on $\dot{b}_{p,q}^s(\mathbb{R}^n)$; see [6, Theorem 3.3] and also [7, Theorem (6.20)]. Here, $\dot{f}_{p,q}^s(\mathbb{R}^n)$ denotes the set of all sequences $\{t_Q\}_{Q \in \mathcal{Q}}$ such that

$$\|\{t_Q\}_{Q \in \mathcal{Q}}\|_{\dot{f}_{p,q}^s(\mathbb{R}^n)} \equiv \left\| \left(\sum_{Q \in \mathcal{Q}} [|Q|^{-s/n-1/2} |t_Q| \chi_Q]^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

and $\dot{b}_{p,q}^s(\mathbb{R}^n)$ the set of all sequences $\{t_Q\}_{Q \in \mathcal{Q}}$ such that

$$\|\{t_Q\}_{Q \in \mathcal{Q}}\|_{\dot{b}_{p,q}^s(\mathbb{R}^n)} \equiv \left\{ \sum_{k \in \mathbb{Z}} \left\| \sum_{Q \in \mathcal{Q}, \ell(Q)=2^{-k}} [|Q|^{-s/n-1/2} |t_Q| \chi_Q] \right\|_{L^p(\mathbb{R}^n)}^q \right\}^{1/q} < \infty.$$

Moreover, by [6, Theorem 2.2] or [7, Theorem (6.16)], $\|u\|_{\dot{F}_{p,q}^s(\mathbb{R}^n)} \sim \|\{t_Q\}_{Q \in \mathcal{Q}}\|_{\dot{f}_{p,q}^s(\mathbb{R}^n)}$, which then implies that

$$\|\tilde{u}\|_{\mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{Q \in \mathcal{Q}} a_{RQ} t_Q \right\}_{R \in \mathcal{Q}} \right\|_{\dot{f}_{p,q}^s(\mathbb{R}^n)} \lesssim \|\{t_Q\}_{Q \in \mathcal{Q}}\|_{\dot{f}_{p,q}^s(\mathbb{R}^n)} \sim \|u\|_{\dot{F}_{p,q}^s(\mathbb{R}^n)}.$$

This argument still holds with the spaces \dot{F} replaced by \dot{B} due to the equivalence that $\|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} \sim \|\{t_Q\}_{Q \in \mathcal{Q}}\|_{\dot{b}_{p,q}^s(\mathbb{R}^n)}$ given by [5, (1.11)]. This finishes the proof of Theorem 3.1. \square

Proof of Theorem 3.2. Observe that with the aid of Theorem 3.1, (iii) and (iv) of Theorem 3.2 imply (i) and (ii) of Theorem 3.2. So it suffices to prove (iii) and (iv) of Theorem 3.2.

We first prove Theorem 3.2(iii), namely, $\dot{M}_{p,q}^s(\mathbb{R}^n) = \mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n)$. To prove $\dot{M}_{p,q}^s(\mathbb{R}^n) \subset \mathcal{A}\dot{F}_{p,q}^s(\mathbb{R}^n)$, let $u \in \dot{M}_{p,q}^s(\mathbb{R}^n)$ and choose $\tilde{g} \in \mathbb{D}^s(u)$ such that $\|\tilde{g}\|_{L^p(\mathbb{R}^n, \ell^q)} \lesssim \|u\|_{\dot{M}_{p,q}^s(\mathbb{R}^n)}$. Then for all $\phi \in \mathcal{A}$, $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$,

$$|\phi_{2^{-k}} * u(x)| = \left| \int_{\mathbb{R}^n} \phi_{2^{-k}}(x-y) [u(y) - u_{B(x, 2^{-k})}] dy \right|$$

$$\lesssim \sum_{j=0}^{\infty} 2^{-2js} \int_{B(x, 2^{-k+j})} |u(y) - u_{B(x, 2^{-k})}| dy.$$

Since

$$\int_{B(x, 2^{-k+j})} |u(y) - u_{B(x, 2^{-k})}| dy \lesssim \sum_{i=0}^j \int_{B(x, 2^{-k+i})} |u(y) - u_{B(x, 2^{-k+i})}| dy,$$

we then have

$$(3.4) \quad |\phi_{2^{-k}} * u(x)| \lesssim \sum_{j=0}^{\infty} 2^{-2js} \int_{B(x, 2^{-k+j})} |u(y) - u_{B(x, 2^{-k+j})}| dy.$$

If $p, q \in (1, \infty]$, then by Lemma 2.1, we have

$$(3.5) \quad \begin{aligned} |\phi_{2^{-k}} * u(x)| &\lesssim \sum_{j=0}^{\infty} 2^{-2js} 2^{-ks+j} \sum_{i=k-j-3}^{k-j} \int_{B(x, 2^{-k+j+2})} g_i(y) dy \\ &\lesssim 2^{-2ks} \sum_{j=-\infty}^k 2^{js} \int_{B(x, 2^{-j+2})} g_j(z) dz \lesssim 2^{-2ks} \sum_{j=-\infty}^k 2^{js} \mathcal{M}(g_j)(x), \end{aligned}$$

where and in what follows, \mathcal{M} denotes the *Hardy-Littlewood maximal function*.

Thus, for $p, q \in (1, \infty)$, by the Hölder inequality and the Fefferman-Stein vector-valued maximal inequality on \mathcal{M} (see [4]), we have

$$(3.6) \quad \begin{aligned} \|u\|_{\dot{A}_{p,q}^s(\mathbb{R}^n)} &\lesssim \left\| \left(\sum_{k \in \mathbb{Z}} 2^{-ksq} \left[\sum_{j=-\infty}^k 2^{js} \mathcal{M}(g_j) \right]^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \left\| \left(\sum_{k \in \mathbb{Z}} 2^{-ks} \sum_{j=-\infty}^k 2^{js} [\mathcal{M}(g_j)]^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \left\| \left(\sum_{j \in \mathbb{Z}} [\mathcal{M}(g_j)]^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|\vec{g}\|_{L^p(\mathbb{R}^n, \ell^q)} \lesssim \|u\|_{\dot{M}_{p,q}^s(\mathbb{R}^n)}. \end{aligned}$$

If $p \in (n/(n+s), 1]$ or $q \in (n/(n+s), 1]$, by (3.4) and Lemma 2.3, choosing $\epsilon, \epsilon' \in (0, s)$ such that $\epsilon < \epsilon'$ and $n/(n+\epsilon') < \min\{p, q\}$, for all $x \in \mathbb{R}^n$,

$$(3.7) \quad \begin{aligned} |\phi_{2^{-k}} * u(x)| &\lesssim \sum_{j=0}^{\infty} 2^{-2js} 2^{-(k-j)\epsilon'} \sum_{i \geq k-j-2} 2^{-i(s-\epsilon')} \\ &\quad \times \left\{ \int_{B(x, 2^{-(k-j)+1})} [g_i(y)]^{n/(n+\epsilon)} dy \right\}^{(n+\epsilon)/n} \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{i=-\infty}^{k-2} \sum_{j \geq k-i-2} 2^{-2js} 2^{-(k-j)\epsilon'} 2^{-i(s-\epsilon')} \\
&\quad \times \left\{ \int_{B(x, 2^{-(k-j)+1})} [g_i(y)]^{n/(n+\epsilon)} dy \right\}^{(n+\epsilon)/n} \\
&\lesssim 2^{-2sk} \sum_{i=-\infty}^{k-2} 2^{is} \left[\mathcal{M}([g_i]^{n/(n+\epsilon)})(x) \right]^{(n+\epsilon)/n}.
\end{aligned}$$

Thus, for $p, q \in (n/(n+s), \infty)$ and $\min\{p, q\} \in (n/(n+s), 1]$, by the Hölder inequality when $q \in (1, \infty)$, (2.2) when $q \in (n/(n+s), 1]$ and the Fefferman-Stein vector-valued maximal inequality, similarly to (3.6), we obtain

$$\begin{aligned}
\|u\|_{\dot{\mathcal{A}}_{p,q}^s(\mathbb{R}^n)} &\lesssim \left\| \left(\sum_{k \in \mathbb{Z}} 2^{-ksq} \left[\sum_{j=-\infty}^{k-2} 2^{js} \left[\mathcal{M}([g_j]^{n/(n+\epsilon)}) \right]^{(n+\epsilon)/n} \right]^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \left\| \left(\sum_{j \in \mathbb{Z}} \left[\mathcal{M}([g_j]^{n/(n+\epsilon)}) \right]^{(n+\epsilon)q/n} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|\vec{g}\|_{L^p(\mathbb{R}^n, \ell^q)} \lesssim \|u\|_{\dot{M}_{p,q}^s(\mathbb{R}^n)}.
\end{aligned}$$

If $p \in (n/(n+s), \infty)$ and $q = \infty$, by (3.5), (3.7), the Fefferman-Stein vector-valued maximal inequality and an argument similar to (3.6), we have $\|u\|_{\dot{\mathcal{A}}_{p,q}^s(\mathbb{R}^n)} \lesssim \|u\|_{\dot{M}_{p,q}^s(\mathbb{R}^n)}$.

If $p = \infty$ and $q \in (1, \infty)$, then for all $x \in \mathbb{R}^n$ and all $\ell \in \mathbb{Z}$, by the Hölder inequality and (3.5), we have that

$$\begin{aligned}
&\int_{B(x, 2^{-\ell})} \sum_{k \geq \ell} 2^{ksq} \sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u(z)|^q dz \\
&\lesssim \int_{B(x, 2^{-\ell})} \sum_{k \geq \ell} 2^{-ksq} \left[\sum_{j=-\infty}^k 2^{js} \int_{B(z, 2^{-j+2})} g_j(y) dy \right]^q dz \\
&\lesssim \int_{B(x, 2^{-\ell})} \sum_{k \geq \ell} 2^{-ks} \sum_{j=-\infty}^k 2^{js} \left[\int_{B(z, 2^{-j+2})} g_j(y) dy \right]^q dz.
\end{aligned}$$

We continue to estimate the last quantity by dividing $\sum_{j=-\infty}^k$ into $\sum_{j=-\infty}^{\ell}$ and $\sum_{j=\ell+1}^k$ when $k > \ell$. Notice that for all $z \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, by the Hölder inequality, we obtain

$$(3.8) \quad \left[\int_{B(z, 2^{-j+2})} g_j(y) dy \right]^q \leq \int_{B(z, 2^{-j+2})} [g_j(y)]^q dy \leq \|u\|_{\dot{M}_{\infty,q}^s(\mathbb{R}^n)}^q.$$

From this, it follows that

$$\int_{B(x, 2^{-\ell})} \sum_{k \geq \ell} 2^{-ks} \sum_{j=-\infty}^{\ell} 2^{js} \left[\int_{B(z, 2^{-j+2})} g_j(y) dy \right]^q dz \lesssim \|u\|_{\dot{M}_{\infty,q}^s(\mathbb{R}^n)}^q.$$

Moreover, since $B(z, 2^{-j+2}) \subset B(x, 2^{-\ell+2})$ for all $j \geq \ell + 1$ and all $z \in B(x, 2^{-\ell})$, by the $L^q(\mathbb{R}^n)$ -boundedness of \mathcal{M} , we obtain that

$$\begin{aligned} & \int_{B(x, 2^{-\ell})} \sum_{k \geq \ell} 2^{-ks} \sum_{j=\ell+1}^k 2^{js} \left[\int_{B(z, 2^{-j+2})} g_j(y) dy \right]^q dz \\ & \lesssim \int_{B(x, 2^{-\ell})} \sum_{k \geq \ell} 2^{-ks} \sum_{j=\ell+1}^k 2^{js} \left[\mathcal{M}(g_j \chi_{B(x, 2^{-\ell+2})})(z) \right]^q dz \\ & \lesssim \sum_{j=\ell+1}^{\infty} \int_{B(x, 2^{-\ell})} \left[\mathcal{M}(g_j \chi_{B(x, 2^{-\ell+2})})(z) \right]^q dz \\ & \lesssim \sum_{j=\ell+1}^{\infty} \int_{B(x, 2^{-\ell+2})} [g_j(y)]^q dy \lesssim \|u\|_{\dot{M}_{\infty, q}^s(\mathbb{R}^n)}^q. \end{aligned}$$

Thus, $\|u\|_{\dot{A}_{\infty, q}^s(\mathbb{R}^n)} \lesssim \|u\|_{\dot{M}_{\infty, q}^s(\mathbb{R}^n)}$.

If $p = \infty$ and $q = \infty$, then the proof is similar but easier than the case $p = \infty$ and $q \in (0, \infty)$. We omit the details.

If $p = \infty$ and $q \in (n/(n+s), 1]$, then from (3.7) with $\epsilon \in (0, s)$ satisfying that $n/(n+\epsilon) < q$, it follows that

$$\begin{aligned} & \int_{B(x, 2^{-\ell})} \sum_{k \geq \ell} 2^{ksq} \sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u(z)|^q dz \\ & \lesssim \int_{B(x, 2^{-\ell})} \sum_{k \geq \ell} 2^{ksq} \sum_{j=0}^{\infty} 2^{-2jsq} 2^{-(k-j)\epsilon q} \sum_{i \geq k-j-2} 2^{-i(s-\epsilon)q} \\ & \quad \times \left\{ \int_{B(x, 2^{-(k-j)+1})} [g_i(y)]^{n/(n+\epsilon)} dy \right\}^{(n+\epsilon)q/n} dz \\ & \lesssim \int_{B(x, 2^{-\ell})} \sum_{j=-\infty}^{\infty} 2^{-sq \max\{\ell, j\}} 2^{2jsq} 2^{-j\epsilon q} \sum_{i \geq j-2} 2^{-i(s-\epsilon)q} \\ & \quad \times \left\{ \int_{B(x, 2^{-j+1})} [g_i(y)]^{n/(n+\epsilon)} dy \right\}^{(n+\epsilon)q/n} dz. \end{aligned}$$

Notice that similarly to (3.8), for $i \geq j-1$ and $x \in \mathbb{R}^n$, by the Hölder inequality and $q > n/(n+\epsilon)$, we have

$$\left\{ \int_{B(x, 2^{-j+1})} [g_i(y)]^{n/(n+\epsilon)} dy \right\}^{(n+\epsilon)/n} \leq \left\{ \int_{B(x, 2^{-j+1})} [g_i(y)]^q dy \right\}^{1/q} \leq \|u\|_{\dot{M}_{\infty, q}^s(\mathbb{R}^n)},$$

which implies that

$$\int_{B(x, 2^{-\ell})} \sum_{j=-\infty}^{\ell} 2^{-\ell sq} 2^{2jsq} 2^{-j\epsilon q} \sum_{i \geq j-2} 2^{-i(s-\epsilon)q}$$

$$\begin{aligned} & \times \left\{ \int_{B(x, 2^{-j+1})} [g_i(y)]^{n/(n+\epsilon)} dy \right\}^{(n+\epsilon)q/n} dz \\ & \lesssim \|u\|_{\dot{M}_{\infty, q}^s(\mathbb{R}^n)}^q \sum_{j=-\infty}^{\ell} 2^{-\ell s q} 2^{j s q} \lesssim \|u\|_{\dot{M}_{\infty, q}^s(\mathbb{R}^n)}^q. \end{aligned}$$

On the other hand, from $(n+\epsilon)q/n > 1$ and the $L^{q(n+\epsilon)/n}(\mathbb{R}^n)$ -boundedness of \mathcal{M} , it follows that

$$\begin{aligned} & \int_{B(x, 2^{-\ell})} \sum_{j=\ell+1}^{\infty} 2^{j s q} 2^{-j \epsilon q} \sum_{i \geq j-2} 2^{-i(s-\epsilon)q} \left\{ \int_{B(x, 2^{-j+1})} [g_i(y)]^{n/(n+\epsilon)} dy \right\}^{(n+\epsilon)q/n} dz \\ & \lesssim \int_{B(x, 2^{-\ell})} \sum_{i \geq \ell-1} \left[\mathcal{M}([g_i]^{n/(n+\epsilon)} \chi_{B(x, 2^{-\ell})})(z) \right]^{(n+\epsilon)q/n} dz \\ & \lesssim \int_{B(x, 2^{-\ell})} \sum_{i \geq \ell-1} [g_i(z)]^q dz \lesssim \|u\|_{\dot{M}_{\infty, q}^s(\mathbb{R}^n)}^q, \end{aligned}$$

which implies that $\|u\|_{\mathcal{A}\dot{F}_{\infty, q}^s(\mathbb{R}^n)} \lesssim \|u\|_{\dot{M}_{\infty, q}^s(\mathbb{R}^n)}$. We have completed the proof of that $\dot{M}_{p, q}^s(\mathbb{R}^n) \subset \mathcal{A}\dot{F}_{p, q}^s(\mathbb{R}^n)$.

To prove $\mathcal{A}\dot{F}_{p, q}^s(\mathbb{R}^n) \subset \dot{M}_{p, q}^s(\mathbb{R}^n)$, let $u \in \mathcal{A}\dot{F}_{p, q}^s(\mathbb{R}^n)$. Since $\mathcal{A}\dot{F}_{p, q}^s(\mathbb{R}^n) \subset \mathcal{A}\dot{F}_{p, \infty}^s(\mathbb{R}^n) = \dot{M}^{s, p}(\mathbb{R}^n) \subset L_{\text{loc}}^1(\mathbb{R}^n)$ by Proposition 2.1 and Lemma 2.1 together with [22, Corollary 2.1], we know that $u \in L_{\text{loc}}^1(\mathbb{R}^n)$. Fix $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with compact support and $\int_{\mathbb{R}^n} \varphi(x) dx = 1$. Notice that $\varphi_{2^{-k}} * u(x) \rightarrow u(x)$ as $k \rightarrow \infty$ for almost all $x \in \mathbb{R}^n$. Then for almost all $x, y \in \mathbb{R}^n$, letting $k_0 \in \mathbb{Z}$ such that $2^{-k_0-1} \leq |x-y| < 2^{-k_0}$, we have

$$\begin{aligned} |u(x) - u(y)| & \leq |\varphi_{2^{-k_0}} * u(x) - \varphi_{2^{-k_0}} * u(y)| \\ & \quad + \sum_{k \geq k_0} (|\varphi_{2^{-k-1}} * u(x) - \varphi_{2^{-k}} * u(x)| + |\varphi_{2^{-k-1}} * u(y) - \varphi_{2^{-k}} * u(y)|). \end{aligned}$$

Write $\varphi_{2^{-k_0}} * u(x) - \varphi_{2^{-k_0}} * u(y) = (\phi^{(x, y)})_{2^{-k_0}} * f(x)$ with $\phi^{(x, y)}(z) \equiv \varphi(z - 2^{k_0}[x-y]) - \varphi(z)$ and $\varphi_{2^{-k-1}} * u(x) - \varphi_{2^{-k}} * u(x) = (\varphi_{2^{-1}} - \varphi)_{2^{-k}} * u(x)$. Notice that $\varphi_{2^{-1}} - \varphi$ and $\phi^{(x, y)}$ are fixed constant multiples of elements of \mathcal{A} . For all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, set

$$(3.9) \quad g_k(x) \equiv 2^{ks} \sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u(x)|.$$

Then we have

$$|u(x) - u(y)| \lesssim \sum_{k \geq k_0} 2^{-ks} [g_k(x) + g_k(y)],$$

which means that $\vec{g} \equiv \{g_k\}_{k \in \mathbb{Z}} \in \tilde{\mathbb{D}}^{s, s, 0}(u)$.

Thus, if $p \in (n/(n+s), \infty)$, then Theorem 2.1 implies that

$$\|u\|_{\dot{M}_{p, q}^s(\mathbb{R}^n)} \lesssim \|\vec{g}\|_{L^p(\mathbb{R}^n; \ell^q)} \lesssim \left\| \left(\sum_{j \in \mathbb{Z}} 2^{j s q} \sup_{\phi \in \mathcal{A}} |\phi_{2^{-j}} * u|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|u\|_{\mathcal{A}\dot{F}_{p, q}^s(\mathbb{R}^n)}.$$

If $p = \infty$ and $q \in (n/(n+s), \infty)$, then by Theorem 2.2, we obtain

$$\int_{B(x, 2^{-\ell})} \sum_{k \geq \ell} [g_k(y)]^q dy \lesssim \int_{B(x, 2^{-\ell})} \sum_{k \geq \ell} 2^{ksq} \sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u(y)|^q dy \lesssim \|u\|_{\dot{\mathcal{A}}_{\infty, q}^s(\mathbb{R}^n)},$$

for all $x \in \mathbb{R}^n$ and $\ell \in \mathbb{Z}$. Thus, $\|u\|_{\dot{M}_{\infty, q}^s(\mathbb{R}^n)} \lesssim \|u\|_{\dot{\mathcal{A}}_{\infty, q}^s(\mathbb{R}^n)}$.

If $p = \infty$ and $q = \infty$, the proof is similar and easier. We omit the details. This finishes the proof of Theorem 3.2(iii).

Now, we prove Theorem 3.2(iv), namely, $\dot{N}_{p, q}^s(\mathbb{R}^n) = \dot{\mathcal{A}}_{p, q}^s(\mathbb{R}^n)$. To prove $\dot{N}_{p, q}^s(\mathbb{R}^n) \subset \dot{\mathcal{A}}_{p, q}^s(\mathbb{R}^n)$, let $\epsilon \in (0, s)$ such that $n/(n+\epsilon) < p$ and notice that (3.7) still holds here. Then for all $u \in \dot{N}_{p, q}^s(\mathbb{R}^n)$ and $\vec{g} \in \mathbb{D}^s(u)$, by (3.7), we have

$$\|u\|_{\dot{\mathcal{A}}_{p, q}^s(\mathbb{R}^n)} \lesssim \left(\sum_{k \in \mathbb{Z}} 2^{-ksq} \left\| \sum_{j=-\infty}^{k-2} 2^{js} [\mathcal{M}([g_j]^{n/(n+\epsilon)})]^{(n+\epsilon)/n} \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q}.$$

Now we consider two cases. If $p \in (n/(n+\epsilon), 1]$, by (2.2) with q there replaced by p , we further obtain

$$\|u\|_{\dot{\mathcal{A}}_{p, q}^s(\mathbb{R}^n)} \lesssim \left\{ \sum_{k \in \mathbb{Z}} 2^{-ksq} \left(\sum_{j=-\infty}^{k-2} 2^{jsp} \left\| [\mathcal{M}([g_j]^{n/(n+\epsilon)})]^{(n+\epsilon)/n} \right\|_{L^p(\mathbb{R}^n)}^p \right)^{q/p} \right\}^{1/q}.$$

From this, the Hölder inequality when $q > p$ and (2.2) with q there replaced by q/p when $q \leq p$, and the $L^{p(n+\epsilon)/n}(\mathbb{R}^n)$ -boundedness of \mathcal{M} , it follows that

$$\begin{aligned} \|u\|_{\dot{\mathcal{A}}_{p, q}^s(\mathbb{R}^n)} &\lesssim \left(\sum_{k \in \mathbb{Z}} 2^{-ksq/2} \sum_{j=-\infty}^{k-2} 2^{jsq/2} \|g_j\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \\ &\sim \left(\sum_{j \in \mathbb{Z}} \|g_j\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \sim \|\vec{g}\|_{\ell^q(L^p(\mathbb{R}^n))} \lesssim \|u\|_{\dot{N}_{p, q}^s(\mathbb{R}^n)}. \end{aligned}$$

If $p \in (1, \infty]$, then by the Minkowski inequality, we have

$$\|u\|_{\dot{\mathcal{A}}_{p, q}^s(\mathbb{R}^n)} \lesssim \left\{ \sum_{k \in \mathbb{Z}} 2^{-ksq} \left(\sum_{j=-\infty}^k 2^{js} \left\| [\mathcal{M}([g_j]^{n/(n+\epsilon)})]^{(n+\epsilon)/n} \right\|_{L^p(\mathbb{R}^n)} \right)^q \right\}^{1/q},$$

which together with the Hölder inequality or (2.2) when $q \in (0, 1]$, and the $L^{p(n+\epsilon)/n}(\mathbb{R}^n)$ -boundedness of \mathcal{M} also yields that

$$\|u\|_{\dot{\mathcal{A}}_{p, q}^s(\mathbb{R}^n)} \lesssim \left(\sum_{k \in \mathbb{Z}} 2^{-ksq/2} \sum_{j=-\infty}^{k-2} 2^{jsq/2} \|g_j\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \lesssim \|u\|_{\dot{N}_{p, q}^s(\mathbb{R}^n)}.$$

Thus, $\dot{N}_{p,q}^s(\mathbb{R}^n) \subset \dot{\mathcal{A}}\dot{B}_{p,q}^s(\mathbb{R}^n)$.

Conversely, to show $\dot{\mathcal{A}}\dot{B}_{p,q}^s(\mathbb{R}^n) \subset \dot{N}_{p,q}^s(\mathbb{R}^n)$, let $u \in \dot{\mathcal{A}}\dot{B}_{p,q}^s(\mathbb{R}^n)$. Then we claim that $u \in L_{\text{loc}}^1(\mathbb{R}^n)$. Assume that this claim holds for the moment. Taking $\vec{g} \equiv \{g_k\}_{k \in \mathbb{Z}}$ with g_k as in (3.9) and by an argument similar to the proof of $\dot{\mathcal{A}}\dot{B}_{p,q}^s(\mathbb{R}^n) \subset \dot{M}_{p,q}^s(\mathbb{R}^n)$, we know that $\vec{g} \in \widetilde{\mathbb{D}}^{s,s,0}(u)$. By Theorem 2.1, we have

$$\begin{aligned} \|u\|_{\dot{N}_{p,q}^s(\mathbb{R}^n)} &\lesssim \left(\sum_{j \in \mathbb{Z}} \|g_j\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \\ &\lesssim \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \left\| \sup_{\phi \in \mathcal{A}} |\phi_{2^{-j}} * u| \right\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} \lesssim \|u\|_{\dot{\mathcal{A}}\dot{B}_{p,q}^s(\mathbb{R}^n)}, \end{aligned}$$

which implies that $\dot{\mathcal{A}}\dot{B}_{p,q}^s(\mathbb{R}^n) \subset \dot{N}_{p,q}^s(\mathbb{R}^n)$.

Finally, we prove the above claim that $u \in L_{\text{loc}}^1(\mathbb{R}^n)$. If $p = \infty$, since

$$\dot{\mathcal{A}}\dot{B}_{\infty,q}^s(\mathbb{R}^n) \subset \dot{\mathcal{A}}\dot{B}_{\infty,\infty}^s(\mathbb{R}^n) = \dot{\mathcal{A}}\dot{F}_{\infty,\infty}^s(\mathbb{R}^n) = \dot{M}^{s,\infty}(\mathbb{R}^n) \subset L_{\text{loc}}^1(\mathbb{R}^n)$$

by [22, Corollary 1.2], then $u \in L_{\text{loc}}^1(\mathbb{R}^n)$. For $p \in (n/(n+s), \infty)$, let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \varphi(z) dz = 1$. Then $\varphi_{2^{-k}} * u \rightarrow u$ in $\mathcal{S}'(\mathbb{R}^n)$ and hence

$$u = \varphi * u + \sum_{k=0}^{\infty} (\varphi_{2^{-k-1}} * u - \varphi_{2^{-k}} * u)$$

in $\mathcal{S}'(\mathbb{R}^n)$. Observe that for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}_+$,

$$|\varphi_{2^{-k-1}} * u(x) - \varphi_{2^{-k}} * u(x)| \lesssim \sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u(x)|.$$

If $p \in [1, \infty)$, then

$$\sum_{k=0}^{\infty} \|\varphi_{2^{-k-1}} * u - \varphi_{2^{-k}} * u\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{k=0}^{\infty} 2^{-ks} \|u\|_{\dot{\mathcal{A}}\dot{B}_{p,q}^s(\mathbb{R}^n)} \lesssim \|u\|_{\dot{\mathcal{A}}\dot{B}_{p,q}^s(\mathbb{R}^n)},$$

which implies that $\sum_{k=0}^{\infty} (\varphi_{2^{-k-1}} * u - \varphi_{2^{-k}} * u)$ converges in $L^p(\mathbb{R}^n)$. Observing that $\varphi * u$ is a continuous function, we know that $\varphi * u + \sum_{k=0}^{\infty} (\varphi_{2^{-k-1}} * u - \varphi_{2^{-k}} * u) \in L_{\text{loc}}^1(\mathbb{R}^n)$, which implies that u is an element of $\mathcal{S}'(\mathbb{R}^n)$ induced by a function in $L_{\text{loc}}^1(\mathbb{R}^n)$. In this sense, we say that $u \in L_{\text{loc}}^1(\mathbb{R}^n)$. For $p \in (n/(n+s), 1)$, it is easy to see that for all $\phi \in \mathcal{A}$, $k \in \mathbb{Z}$, $x \in \mathbb{R}^n$ and $y \in B(x, 2^{-k})$, the function $\tilde{\phi}(z) \equiv \phi(z + 2^k(x - y))$ for all $z \in \mathbb{R}^n$ is a constant multiple of an element of \mathcal{A} with the constant independent of x, y and k . Notice that $\phi_{2^{-k}} * u(x) = \tilde{\phi}_{2^{-k}} * u(y)$. Then for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$\sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u(x)| = \left(\int_{B(x, 2^{-k})} \sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u(x)|^p dy \right)^{1/p}$$

$$\lesssim \left(\int_{B(x, 2^{-k})} \sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u(y)|^p dy \right)^{1/p} \lesssim 2^{kn/p} \left\| \sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u| \right\|_{L^p(\mathbb{R}^n)}$$

and hence

$$\begin{aligned} \left\| \sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u| \right\|_{L^1(\mathbb{R}^n)} &\lesssim 2^{k(1-p)n/p} \left\| \sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u|^p \right\|_{L^1(\mathbb{R}^n)}^{1/p} \left\| \sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u| \right\|_{L^p(\mathbb{R}^n)}^{1-p} \\ &\lesssim 2^{kn(1/p-1)} \left\| \sup_{\phi \in \mathcal{A}} |\phi_{2^{-k}} * u| \right\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

which together with $p > n/(n+s)$ implies that

$$\sum_{k=0}^{\infty} \|\varphi_{2^{-k-1}} * u - \varphi_{2^{-k}} * u\|_{L^1(\mathbb{R}^n)} \lesssim \sum_{k=0}^{\infty} 2^{-k(n+s-n/p)} \|u\|_{\dot{A}_{p,q}^s(\mathbb{R}^n)} \lesssim \|u\|_{\dot{A}_{p,q}^s(\mathbb{R}^n)}.$$

From this and an argument similar to the case $p \in [1, \infty)$, it follows that $u \in L_{\text{loc}}^1(\mathbb{R}^n)$. This shows the above claim and finishes the proof of Theorem 3.2(iv) and hence of Theorem 3.2. \square

4 Besov and Triebel-Lizorkin spaces on RD-spaces

Let (\mathcal{X}, d, μ) be an *RD-space* throughout the whole section. We extend Theorems 3.1 and 3.2 to the Besov and Triebel-Lizorkin spaces on \mathcal{X} ; see Theorem 4.1. We also establish an equivalence of $\dot{M}_{p,p}^s(\mathcal{X})$ and the Besov space $\dot{B}_p^s(\mathcal{X})$ considered by Bourdon and Pajot [2]; see Proposition 4.1.

We begin with the definition of the homogeneous (grand) Besov and Triebel-Lizorkin spaces on RD-spaces. To this end, we first recall the spaces of test functions on RD-spaces; see [14]. For our convenience, in what follows, for any $x, y \in \mathcal{X}$ and $r > 0$, we always set $V(x, y) \equiv \mu(B(x, d(x, y)))$ and $V_r(x) \equiv \mu(B(x, r))$. It is easy to see that $V(x, y) \sim V(y, x)$ for all $x, y \in \mathcal{X}$. Moreover, if $\mu(\mathcal{X}) < \infty$, then $\text{diam } \mathcal{X} < \infty$ and hence without loss of generality, we may always assume that $\text{diam } \mathcal{X} = 2^{-k_0}$ for some $k_0 \in \mathbb{Z}$.

Definition 4.1. Let $x_1 \in \mathcal{X}$, $r \in (0, \infty)$, $\beta \in (0, 1]$ and $\gamma \in (0, \infty)$. A function φ on \mathcal{X} is said to be in the space $\mathcal{G}(x_1, r, \beta, \gamma)$ if there exists a nonnegative constant C such that

- (i) $|\varphi(x)| \leq C \frac{1}{V_r(x_1) + V(x_1, x)} \left(\frac{r}{r + d(x_1, x)} \right)^\gamma$ for all $x \in \mathcal{X}$;
- (ii) $|\varphi(x) - \varphi(y)| \leq C \left(\frac{d(x, y)}{r + d(x_1, x)} \right)^\beta \frac{1}{V_r(x_1) + V(x_1, x)} \left(\frac{r}{r + d(x_1, x)} \right)^\gamma$ for all $x, y \in \mathcal{X}$ satisfying that $d(x, y) \leq (r + d(x_1, x))/2$.

Moreover, for any $\varphi \in \mathcal{G}(x_1, r, \beta, \gamma)$, its *norm* is defined by

$$\|\varphi\|_{\mathcal{G}(x_1, r, \beta, \gamma)} \equiv \inf\{C : (i) \text{ and } (ii) \text{ hold}\}.$$

Throughout this section, we fix $x_1 \in \mathcal{X}$ and let $\mathcal{G}(\beta, \gamma) \equiv \mathcal{G}(x_1, 1, \beta, \gamma)$. Then $\mathcal{G}(\beta, \gamma)$ is a Banach space. We also let $\dot{\mathcal{G}}(\beta, \gamma) \equiv \{f \in \mathcal{G}(\beta, \gamma) : \int_{\mathcal{X}} f(x) d\mu(x) = 0\}$. Denote by $(\mathcal{G}(\beta, \gamma))'$ and $(\dot{\mathcal{G}}(\beta, \gamma))'$ the *dual spaces* of $\mathcal{G}(\beta, \gamma)$ and $\dot{\mathcal{G}}(\beta, \gamma)$, respectively. Obviously, $(\dot{\mathcal{G}}(\beta, \gamma))' = (\mathcal{G}(\beta, \gamma))'/\mathbb{C}$.

Let $\epsilon \in (0, 1]$ and $\beta, \gamma \in (0, \epsilon)$. Define $\mathcal{G}_0^\epsilon(\beta, \gamma)$ as the *completion of the set* $\mathcal{G}(\epsilon, \epsilon)$ in the space $\mathcal{G}(\beta, \gamma)$, and for $\varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma)$, define $\|\varphi\|_{\mathcal{G}_0^\epsilon(\beta, \gamma)} \equiv \|\varphi\|_{\mathcal{G}(\beta, \gamma)}$. Then, it is easy to see that $\mathcal{G}_0^\epsilon(\beta, \gamma)$ is a Banach space. Similarly, we define $\dot{\mathcal{G}}_0^\epsilon(\beta, \gamma)$. Let $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$ and $(\dot{\mathcal{G}}_0^\epsilon(\beta, \gamma))'$ be the *dual spaces* of $\mathcal{G}_0^\epsilon(\beta, \gamma)$ and $\dot{\mathcal{G}}_0^\epsilon(\beta, \gamma)$, respectively. Obviously, $(\dot{\mathcal{G}}_0^\epsilon(\beta, \gamma))' = (\mathcal{G}_0^\epsilon(\beta, \gamma))'/\mathbb{C}$.

Now we recall the notion of approximations of the identity on RD-spaces, which were first introduced in [14].

Definition 4.2. Let $\epsilon_1 \in (0, 1]$ and assume that $\mu(\mathcal{X}) = \infty$. A sequence $\{S_k\}_{k \in \mathbb{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is called an *approximation of the identity of order ϵ_1 with bounded support* (for short, ϵ_1 -AOTI with bounded support), if there exist positive constants C_3 and C_4 such that for all $k \in \mathbb{Z}$ and all x, x', y and $y' \in \mathcal{X}$, $S_k(x, y)$, the integral kernel of S_k is a measurable function from $\mathcal{X} \times \mathcal{X}$ into \mathbb{C} satisfying

- (i) $S_k(x, y) = 0$ if $d(x, y) > C_4 2^{-k}$ and $|S_k(x, y)| \leq C_3 \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}$;
- (ii) $|S_k(x, y) - S_k(x', y)| \leq C_3 2^{k\epsilon_1} d(x, x')^{\epsilon_1} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}$ for $d(x, x') \leq \max\{C_4, 1\} 2^{1-k}$;
- (iii) Property (ii) holds with x and y interchanged;
- (iv) $|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]| \leq C_3 2^{2k\epsilon_1} \frac{[d(x, x')]^{\epsilon_1} [d(y, y')]^{\epsilon_1}}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}$ for $d(x, x') \leq \max\{C_4, 1\} 2^{1-k}$ and $d(y, y') \leq \max\{C_4, 1\} 2^{1-k}$;
- (v) $\int_{\mathcal{X}} S_k(x, z) d\mu(z) = 1 = \int_{\mathcal{X}} S_k(z, y) d\mu(z)$.

Remark 4.1. It was proved in [14, Theorem 2.6] that there always exists a 1-AOTI with bounded support on RD-spaces.

Recall the notion of homogeneous Triebel-Lizorkin spaces in [14] as follows.

Definition 4.3. Let $\epsilon \in (0, 1)$, $s \in (0, \epsilon)$ and $p \in (n/(n + \epsilon), \infty]$. Let $\beta, \gamma \in (0, \epsilon)$ such that $\beta \in (s, \epsilon)$ and $\gamma \in (\max\{s - \kappa/p, n/p - n, 0\}, \epsilon)$. Assume that $\mu(\mathcal{X}) = \infty$ and $\{S_k\}_{k \in \mathbb{Z}}$ is an ϵ -AOTI with bounded support as in Definition 4.2. For $k \in \mathbb{Z}$, set $D_k \equiv S_k - S_{k-1}$.

(i) Let $q \in (n/(n + \epsilon), \infty]$. The *homogeneous Triebel-Lizorkin space* $\dot{F}_{p, q}^s(\mathcal{X})$ is defined to be the set of all $f \in (\dot{\mathcal{G}}_0^\epsilon(\beta, \gamma))'$ such that $\|f\|_{\dot{F}_{p, q}^s(\mathcal{X})} < \infty$, where when $p \in (n/(n + \epsilon), \infty)$,

$$(4.1) \quad \|f\|_{\dot{F}_{p, q}^s(\mathcal{X})} \equiv \left\| \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |D_k(f)|^q \right\}^{1/q} \right\|_{L^p(\mathcal{X})}$$

with the usual modification made when $q = \infty$, while when $p = \infty$,

$$(4.2) \quad \|f\|_{\dot{F}_{\infty, q}^s(\mathcal{X})} \equiv \sup_{x \in \mathcal{X}} \sup_{\ell \in \mathbb{Z}} \left(\int_{B(x, 2^{-\ell})} \sum_{k \geq \ell} 2^{ksq} |D_k(f)(y)|^q d\mu(y) \right)^{1/q}$$

with the usual modification made when $q = \infty$.

(ii) Let $q \in (0, \infty]$. The *homogeneous Besov space* $\dot{B}_{p,q}^s(\mathcal{X})$ is defined to be the set of all $f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma))'$ such that

$$(4.3) \quad \|f\|_{\dot{B}_{p,q}^s(\mathcal{X})} \equiv \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} \|D_k(f)\|_{L^p(\mathcal{X})}^q \right\}^{1/q} < \infty$$

with the usual modification made when $q = \infty$.

Remark 4.2. (i) As shown in [32], the definition of $\dot{F}_{p,q}^s(\mathcal{X})$ is independent of the choices of ϵ, β, γ and the approximation of the identity as in Definition 4.2.

(ii) By $(\mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma))' = (\mathcal{G}_0^\epsilon(\beta, \gamma))'/\mathbb{C}$, if we replace $(\mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma))'$ with $(\mathcal{G}_0^\epsilon(\beta, \gamma))'/\mathbb{C}$ or similarly with $(\mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma))'$ in Definition 4.3, then we obtain new Besov and Triebel-Lizorkin spaces, which modulo constants are equivalent to the original Besov and Triebel-Lizorkin spaces respectively. So we can replace $(\mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma))'$ with $(\mathcal{G}_0^\epsilon(\beta, \gamma))'/\mathbb{C}$ or $(\mathring{\mathcal{G}}_0^\epsilon(\beta, \gamma))'$ in the Definition 4.3 if need be, in what follows.

To define grand Besov and Triebel-Lizorkin spaces, we introduce the class of test functions. Motivated by [22], when $\mu(\mathcal{X}) = \infty$, for all $x \in \mathcal{X}$ and $k \in \mathbb{Z}$, let

$$(4.4) \quad \mathcal{A}_k(x) \equiv \{\phi \in \mathring{\mathcal{G}}(1, 2) : \|\phi\|_{\mathring{\mathcal{G}}(x, 2^{-k}, 1, 2)} \leq 1\};$$

when $\mu(\mathcal{X}) = 2^{-k_0}$, for all $x \in \mathcal{X}$ and $k \geq k_0$, let $\mathcal{A}_k(x)$ be as in (4.4), and for $k < k_0$, let $\mathcal{A}_k(x) \equiv \{0\}$. Set $\mathcal{A} \equiv \{\mathcal{A}_k(x)\}_{x \in \mathcal{X}, k \in \mathbb{Z}}$. Moreover, we also introduce the class of test functions with bounded support. For all $x \in \mathcal{X}$ and $k \in \mathbb{Z}$, let

$$(4.5) \quad \tilde{\mathcal{A}}_k(x) \equiv \{\phi \in \mathcal{A}_k(x) : \text{supp } \phi \subset B(x, 2^{-k})\}.$$

Set $\tilde{\mathcal{A}} \equiv \{\tilde{\mathcal{A}}_k(x)\}_{x \in \mathcal{X}, k \in \mathbb{Z}}$.

Definition 4.4. Let $s \in (0, 1]$, $p, q \in (0, \infty]$ and \mathcal{A} be as above.

(i) The *homogeneous grand Triebel-Lizorkin space* $\mathcal{A}\dot{F}_{p,q}^s(\mathcal{X})$ is defined to be the set of all $f \in (\mathcal{G}(1, 2))'$ that satisfy $\|f\|_{\mathcal{A}\dot{F}_{p,q}^s(\mathcal{X})} < \infty$, where $\|f\|_{\mathcal{A}\dot{F}_{p,q}^s(\mathcal{X})}$ is defined as $\|f\|_{\dot{F}_{p,q}^s(\mathcal{X})}$ via replacing $|D_k(f)|$ in (4.1) and (4.2) by $\sup_{\varphi \in \mathcal{A}_k} |\langle f, \varphi \rangle|$.

(ii) The *homogeneous grand Besov space* $\mathcal{A}\dot{B}_{p,q}^s(\mathcal{X})$ is defined to be the set of all $f \in (\mathcal{G}(1, 2))'$ that satisfy $\|f\|_{\mathcal{A}\dot{B}_{p,q}^s(\mathcal{X})} < \infty$, where $\|f\|_{\mathcal{A}\dot{B}_{p,q}^s(\mathcal{X})}$ is defined as $\|f\|_{\dot{B}_{p,q}^s(\mathcal{X})}$ via replacing $|D_k(f)|$ in (4.3) by $\sup_{\varphi \in \mathcal{A}_k} |\langle f, \varphi \rangle|$.

Define the spaces $\tilde{\mathcal{A}}\dot{F}_{p,q}^s(\mathcal{X})$ and $\tilde{\mathcal{A}}\dot{B}_{p,q}^s(\mathcal{X})$ as $\mathcal{A}\dot{F}_{p,q}^s(\mathcal{X})$ and $\mathcal{A}\dot{B}_{p,q}^s(\mathcal{X})$ via replacing \mathcal{A} by $\tilde{\mathcal{A}}$ as in (4.5).

The main result of this section is as follows.

Theorem 4.1. (i) Assume that $\mu(\mathcal{X}) = \infty$. If $s \in (0, 1)$ and $p, q \in (n/(n+s), \infty]$, then $\dot{F}_{p,q}^s(\mathcal{X}) = \dot{M}_{p,q}^s(\mathcal{X})$.

(ii) Assume that $\mu(\mathcal{X}) = \infty$. If $s \in (0, 1)$, $p \in (n/(n+s), \infty]$ and $q \in (0, \infty]$, then $\dot{B}_{p,q}^s(\mathcal{X}) = \dot{N}_{p,q}^s(\mathcal{X})$.

- (iii) If $s \in (0, 1]$ and $p, q \in (n/(n+s), \infty]$, then $\mathcal{A}\dot{F}_{p,q}^s(\mathcal{X}) = \tilde{\mathcal{A}}\dot{F}_{p,q}^s(\mathcal{X}) = \dot{M}_{p,q}^s(\mathcal{X})$.
 (iv) If $s \in (0, 1]$, $p \in (n/(n+s), \infty]$ and $q \in (0, \infty]$, then $\mathcal{A}\dot{B}_{p,q}^s(\mathcal{X}) = \tilde{\mathcal{A}}\dot{B}_{p,q}^s(\mathcal{X}) = \dot{N}_{p,q}^s(\mathcal{X})$.

Proof. The proof of Theorem 4.1 uses the ideas from Theorems 3.1 and 3.2, and also some from [22, Theorems 1.4 and 5.1]. We only point out that all the tools to prove Theorem 4.1 are available. The details are omitted.

Assume that $\mu(\mathcal{X}) = \infty$. Then the result $\mathcal{A}\dot{F}_{p,q}^s(\mathcal{X}) = \dot{F}_{p,q}^s(\mathcal{X})$ for $p < \infty$ is given in [22, Theorem 1.4], whose proof used the discrete Calderón reproducing formula established in [14]. The proofs of $\mathcal{A}\dot{F}_{\infty,q}^s(\mathcal{X}) = \dot{F}_{\infty,q}^s(\mathcal{X})$ and $\mathcal{A}\dot{B}_{p,q}^s(\mathcal{X}) = \dot{B}_{p,q}^s(\mathcal{X})$ can be done by using the discrete Calderón reproducing formula and an argument similar to that used in the proofs of Theorem 3.1 and [22, Theorem 1.4].

The results $\mathcal{A}\dot{F}_{p,q}^s(\mathcal{X}) = \dot{M}_{p,q}^s(\mathcal{X})$ and $\mathcal{A}\dot{B}_{p,q}^s(\mathcal{X}) = \dot{N}_{p,q}^s(\mathcal{X})$ can be proved similarly to the proof of Theorem 3.2. Here we point out that the variants of Lemmas 2.1 through 2.3 still hold in the current setting. In fact, a variant of Lemma 2.2 is given in [22, Lemma 4.1], and variants of Lemmas 2.3 and 2.1 can be proved by using the same ideas as those used in the proof of [22, Lemma 4.1]. Applying these technical lemmas, via an argument as the proofs of (iii) and (iv) of Theorem 3.2, we then obtain $\tilde{\mathcal{A}}\dot{F}_{p,q}^s(\mathcal{X}) = \dot{M}_{p,q}^s(\mathcal{X})$ and $\tilde{\mathcal{A}}\dot{B}_{p,q}^s(\mathcal{X}) = \dot{N}_{p,q}^s(\mathcal{X})$, which completes the proof of Theorem 4.1. \square

Remark 4.3. Assume that $\mu(\mathcal{X}) = 2^{-k_0}$ for some $k_0 \in \mathbb{Z}$. Based on Theorem 4.1, we simply write $\mathcal{A}\dot{B}_{p,q}^s(\mathcal{X})$ as $\dot{B}_{p,q}^s(\mathcal{X})$ and $\mathcal{A}\dot{F}_{p,q}^s(\mathcal{X})$ as $\dot{F}_{p,q}^s(\mathcal{X})$. This is also reasonable in the sense that $\mathcal{A}\dot{B}_{p,p}^s(\mathcal{X})$ and $\mathcal{A}\dot{F}_{p,p}^s(\mathcal{X})$ coincide with $\dot{B}_p^s(\mathcal{X})$ when $s \in (0, 1)$ and $p \in (1, \infty)$; see Proposition 4.1 below. It is still unknown in this case if $\dot{B}_{p,q}^s(\mathcal{X})$ and $\dot{F}_{p,q}^s(\mathcal{X})$ can be characterized via radial Littlewood-Paley functions.

Finally, we establish an equivalence between $\dot{M}_{p,p}^s(\mathcal{X})$ and the Besov space $\dot{B}_p^s(\mathcal{X})$ considered by Bourdon and Pajot [2] as follows. For the characterizations of Besov and Triebel-Lizorkin spaces via differences on metric measure spaces, see [23, 10].

Definition 4.5. Let $s \in (0, \infty)$ and $p \in [1, \infty)$. Denote by $\dot{B}_p^s(\mathcal{X})$ the space of all $u \in L_{\text{loc}}^p(\mathcal{X})$ satisfying that

$$\|u\|_{\dot{B}_p^s(\mathcal{X})} \equiv \left(\int_{\mathcal{X}} \int_{\mathcal{X}} \frac{|u(x) - u(y)|^p}{[d(x, y)]^{sp} V(x, y)} d\mu(y) d\mu(x) \right)^{1/p} < \infty.$$

Proposition 4.1. Let $s \in (0, \infty)$ and $p \in [1, \infty)$. Then $\dot{B}_p^s(\mathcal{X}) = \dot{M}_{p,p}^s(\mathcal{X})$.

Proof. Let $u \in \dot{B}_p^s(\mathcal{X})$. We need to find a fractional s -Hajlasz gradient of u . If $s \in (0, 1]$, we can use the grand maximal function as in the proof of Theorem 1.2, namely, use the equivalence $\dot{M}_{p,p}^s(\mathcal{X}) = \mathcal{A}\dot{F}_{p,p}^s(\mathcal{X})$ given in Theorem 4.1. But, for $s > 1$, we need to find another fractional s -Hajlasz gradient of u . Indeed, we deal with both cases in a uniform way by taking another fractional s -Hajlasz gradient.

By the reverse doubling property of the RD-space, there exists $K_0 \in \mathbb{N}$ and $K_0 > 1$ such that for all $x \in \mathcal{X}$ and $0 < r < 2 \text{diam } \mathcal{X} / 2^{K_0}$, $\mu(B(x, 2^{K_0}r)) \geq 2\mu(B(x, r))$. Notice

that $u \in L_{\text{loc}}^p(\mathcal{X}) \subset L_{\text{loc}}^1(\mathcal{X})$. Thus, for all the Lebesgue points x of u and all $k \in \mathbb{Z}$ such that $2^{-k+K_0} < \text{diam } \mathcal{X}$, by the Hölder inequality, we have that

$$\begin{aligned}
 (4.6) \quad & |u(x) - u_{B(x, 2^{-k})}| \\
 & \leq \sum_{j \geq k} |u_{B(x, 2^{-j})} - u_{B(x, 2^{-j-1})}| \\
 & \lesssim \sum_{j \geq k} \int_{B(x, 2^{-j})} |u(y) - u_{B(x, 2^{-j+K_0+1}) \setminus B(x, 2^{-j+1})}| d\mu(y) \\
 & \lesssim \sum_{j \geq k} \int_{B(x, 2^{-j})} \int_{B(x, 2^{-j+K_0+1}) \setminus B(x, 2^{-j+1})} |u(y) - u(z)| d\mu(z) d\mu(y) \\
 & \lesssim \sum_{j \geq k} 2^{-js} \\
 & \quad \times \left\{ \int_{B(x, 2^{-j})} \int_{B(x, 2^{-j+K_0+1}) \setminus B(x, 2^{-j+1})} \frac{|u(y) - u(z)|^p}{[d(y, z)]^{sp} V(y, z)} d\mu(z) d\mu(y) \right\}^{1/p}.
 \end{aligned}$$

If $\mu(\mathcal{X}) = \infty$, for all $j \in \mathbb{Z}$ and $x \in \mathcal{X}$, we let

$$h_j(x) \equiv \left\{ \int_{B(x, 2^{-j-1})} \int_{B(x, 2^{-j+K_0+1}) \setminus B(x, 2^{-j+1})} \frac{|u(y) - u(z)|^p}{[d(y, z)]^{sp} V(y, z)} d\mu(z) d\mu(y) \right\}^{1/p},$$

and $\vec{h} \equiv \{h_j\}_{j \in \mathbb{Z}}$. Then $\vec{h} \in \widetilde{\mathbb{D}}^{s, s, 1}(u)$. Let y also be a Lebesgue point of u with $2^{-k-1} \leq d(x, y) < 2^{-k}$. Now

$$|u(x) - u(y)| \leq |u(x) - u_{B(x, 2^{-k})}| + |u(y) - u_{B(x, 2^{-k})}|.$$

Observe that by (4.6) and an argument similar to it, we have

$$\begin{aligned}
 |u(y) - u_{B(x, 2^{-k})}| & \leq |u(y) - u_{B(y, 2^{-k+1})}| + |u_{B(y, 2^{-k+1})} - u_{B(x, 2^{-k})}| \\
 & \lesssim \sum_{j \geq k-1} 2^{-js} h_j(y) + \int_{B(y, 2^{-k+1})} |u(z) - u_{B(y, 2^{-k+1})}| dz \\
 & \lesssim \sum_{j \geq k-1} 2^{-js} h_j(y).
 \end{aligned}$$

So

$$(4.7) \quad |u(x) - u(y)| \lesssim \sum_{j \geq k-1} 2^{-js} [h_j(x) + h_j(y)].$$

Thus, by Theorem 2.1, we obtain that

$$\begin{aligned}
 & \|u\|_{\dot{M}_{p, p}^s(\mathcal{X})} \\
 & \sim \int_{\mathcal{X}} \sum_{j \in \mathbb{Z}} \int_{B(x, 2^{-j})} \int_{B(x, 2^{-j+K_0+1}) \setminus B(x, 2^{-j+1})} \frac{|u(y) - u(z)|^p}{[d(y, z)]^{sp} V(y, z)} d\mu(z) d\mu(y) d\mu(x)
 \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{j \in \mathbb{Z}} \int_{\mathcal{X}} \int_{B(y, 2^{-j+K_0+2}) \setminus B(y, 2^{-j})} \frac{|u(y) - u(z)|^p}{[d(y, z)]^{sp} V(y, z)} d\mu(z) d\mu(y) \\
&\lesssim \int_{\mathcal{X}} \int_{\mathcal{X}} \frac{|u(y) - u(z)|^p}{[d(y, z)]^{sp} V(y, z)} d\mu(z) d\mu(y) \sim \|u\|_{\dot{B}_p^s(\mathcal{X})}^p.
\end{aligned}$$

Now assume that $\mu(\mathcal{X}) = 2^{-k_0}$ for some $k_0 \in \mathbb{Z}$. Let x, y be a pair of Lebesgue points of u and assume that $2^{-k-1} \leq d(x, y) < 2^{-k}$ for some $k \geq k_0$. If $k \geq k_0 + K_0 + 1$, then (4.7) still holds. If $k_0 \leq k < k_0 + K_0 + 1$, then

$$\begin{aligned}
(4.8) \quad |u(x) - u(y)| &\lesssim |u(x) - u_{B(x, 2^{-k_0-K_0-3})}| + |u(y) - u_{B(y, 2^{-k_0-K_0-3})}| \\
&\quad + |u_{B(x, 2^{-k_0-K_0-3})} - u_{B(y, 2^{-k_0-K_0-3})}|.
\end{aligned}$$

Since, for all $z \in B(x, 2^{-k_0-K_0-3})$ and $w \in B(y, 2^{-k_0-K_0-3})$, $2^{-k} \gtrsim d(z, w) \geq 2^{-k_0-K_0-3}$, we have that

$$\begin{aligned}
(4.9) \quad &|u_{B(x, 2^{-k_0-K_0-3})} - u_{B(y, 2^{-k_0-K_0-3})}| \\
&\lesssim \int_{B(x, 2^{-k_0-K_0-3})} \int_{B(y, 2^{-k_0-K_0-3})} |u(z) - u(w)| d\mu(z) d\mu(w) \\
&\lesssim 2^{-ks} \left\{ \int_{B(x, 2^{-k_0-K_0-3})} \int_{B(y, 2^{-k_0-K_0-3})} \frac{|u(z) - u(w)|^p}{[d(z, w)]^{sp} V(z, w)} d\mu(z) d\mu(w) \right\}^{1/p} \\
&\lesssim 2^{-ks} [\mu(\mathcal{X})]^{1/p} \|u\|_{\dot{B}_p^s(\mathcal{X})}.
\end{aligned}$$

If we take $h_k \equiv [\mu(\mathcal{X})]^{1/p} \|u\|_{\dot{B}_p^s(\mathcal{X})}$ for all $k_0 - 1 \leq k < k_0 + K_0$ and $h_k \equiv 0$ for $k < k_0 - 1$, then by (4.6), (4.8) and (4.9), we know that (4.7) still holds and hence $\vec{h} \equiv \{h_k\}_{k \in \mathbb{Z}} \in \widetilde{\mathbb{D}}^{s, s, 1}(u)$. Moreover, similarly to the case $\mu(\mathcal{X}) = \infty$, we have $u \in \dot{M}_{p,p}^s(\mathcal{X})$ and

$$\|u\|_{\dot{M}_{p,p}^s(\mathcal{X})} \lesssim \|\vec{h}\|_{L^p(\mathcal{X}, \ell^p)} \lesssim \|u\|_{\dot{B}_p^s(\mathcal{X})}.$$

Conversely, let $u \in \dot{M}_{p,p}^s(\mathcal{X})$. We then have that for all $x \in \mathcal{X}$,

$$\begin{aligned}
\int_{\mathcal{X}} \frac{|u(x) - u(y)|^p}{[d(x, y)]^{sp} V(x, y)} d\mu(y) &\lesssim \sum_{k=k_0}^{\infty} \int_{B(x, 2^{-k}) \setminus B(x, 2^{-k-1})} \frac{|u(x) - u(y)|^p}{[d(x, y)]^{sp} V(x, y)} d\mu(y) \\
&\lesssim \sum_{k=k_0}^{\infty} \int_{B(x, 2^{-k})} [g_k(x) + g_k(y)]^p d\mu(y),
\end{aligned}$$

which implies that $u \in L_{\text{loc}}^p(\mathcal{X})$ and

$$\begin{aligned}
\|u\|_{\dot{B}_p^s(\mathcal{X})}^p &\lesssim \int_{\mathcal{X}} \sum_{k=k_0}^{\infty} \int_{B(x, 2^{-k})} [g_k(x) + g_k(y)]^p d\mu(y) d\mu(x) \\
&\lesssim \int_{\mathcal{X}} \sum_{k=k_0}^{\infty} [g_k(y)]^p d\mu(y) \lesssim \|u\|_{\dot{M}_{p,p}^s(\mathcal{X})}^p.
\end{aligned}$$

This finishes the proof of Proposition 4.1. \square

5 Quasiconformal and quasymmetric mappings

The aim of this section is to prove Theorem 1.3, Theorem 1.4 and their following extension; also see Corollary 5.2.

Theorem 5.1. *Let \mathcal{X} and \mathcal{Y} be Ahlfors n_1 -regular and n_2 -regular spaces with $n_1, n_2 \in (0, \infty)$, respectively. Let f be a quasymmetric mapping from \mathcal{X} onto \mathcal{Y} . For $s_i \in (0, n_i)$ with $i = 1, 2$, if $n_1/s_1 = n_2/s_2$, then f induces an equivalence between $\dot{M}_{n_1/s_1, n_1/s_1}^{s_1}(\mathcal{X})$ and $\dot{M}_{n_2/s_2, n_2/s_2}^{s_2}(\mathcal{Y})$, and hence between $\dot{B}_{n_1/s_1}^{s_1}(\mathcal{X})$ and $\dot{B}_{n_2/s_2}^{s_2}(\mathcal{Y})$.*

Since the volume derivative of a quasymmetric mapping need not satisfy the reverse Hölder inequality in this generality, we cannot extend Theorem 5.1 to the full range $q \in (0, \infty]$. Furthermore, we do not claim that f acts as a composition operator but merely that every $u \in \dot{B}_{n_2/s_2}^{s_2}(\mathcal{Y})$ has a representative \tilde{u} so that $\tilde{u} \circ f \in \dot{B}_{n_1/s_1}^{s_1}(\mathcal{X})$ with a norm bound, and similarly for f^{-1} . Indeed, $u \circ f$ need not even be measurable in this generality.

Now we begin with the proof of Theorem 1.3. To this end, we need the following properties of quasiconformal mappings on \mathbb{R}^n .

First recall that a homeomorphism on \mathbb{R}^n is quasiconformal according to the metric definition if and only if it is quasiconformal according to the analytic definition, and if and only if it is quasymmetric; see, for example, [16, 19]. Moreover, denote by $\mathcal{B}_r(\mathcal{X})$ the class of functions w on the metric measure space \mathcal{X} satisfying the reverse Hölder inequality of order $r \in (1, \infty]$: there exists a positive constant C such that for all balls $B \subset \mathcal{X}$,

$$\left\{ \int_B [w(x)]^r d\mu(x) \right\}^{1/r} \leq C \int_B w(x) d\mu(x).$$

Then, a celebrated result of Gehring [9] says that

Proposition 5.1. *Let $n \geq 2$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a quasiconformal mapping. Then there exists $r \in (1, \infty]$ such that $|J_f| \in \mathcal{B}_r(\mathbb{R}^n)$.*

For a quasiconformal mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we set

$$(5.1) \quad R_f \equiv \sup\{r \in (1, \infty] : |J_f| \in \mathcal{B}_r(\mathbb{R}^n)\}.$$

Notice that $|J_f| \in \mathcal{B}_r(\mathbb{R}^n)$ implies that $|J_f|$ is a weight in the sense of Muckenhoupt. Then, we have the following conclusions; see, for example, [19, Remark 6.1].

Proposition 5.2. *Let $n \geq 2$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a quasiconformal mapping.*

(i) *For any measurable set $E \subset \mathbb{R}^n$, $|f(E)| = \int_E |J_f(x)| dx$; moreover, $|E| = 0$ if and only if $|f(E)| = 0$.*

(ii) *f induces a doubling measure on \mathbb{R}^n , namely, there exists a positive constant C such that for every ball $B \subset \mathbb{R}^n$, $|f(2B)| \leq C|f(B)|$.*

(iii) *There exist positive constants C and $\alpha \in (0, 1]$ such that for every ball $B \subset \mathbb{R}^n$, and every measurable set $E \subset B$,*

$$\frac{|f(E)|}{|f(B)|} \leq C \left(\frac{|E|}{|B|} \right)^\alpha.$$

We also need the following change of variable formula, which is deduced from the Lebesgue-Radon-Nikodym theorem and the absolute continuity of f given in Proposition 5.2(i).

Lemma 5.1. *Let $n \geq 2$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a quasiconformal mapping. Then for all nonnegative Borel measurable functions u on \mathbb{R}^n ,*

$$\int_{\mathbb{R}^n} u(f(x)) |J_f(x)| dx = \int_{\mathbb{R}^n} u(y) dy.$$

Let f be a homeomorphism between metric spaces $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$. For our convenience, in what follows, we always write

$$L_f(x, r) \equiv \sup\{d_{\mathcal{Y}}(f(x), f(y)) : d_{\mathcal{X}}(x, y) \leq r\}$$

and

$$\ell_f(x, r) \equiv \inf\{d_{\mathcal{Y}}(f(x), f(y)) : d_{\mathcal{X}}(x, y) \geq r\}$$

for all $x \in \mathcal{X}$ and $r \in (0, \infty)$.

Proof of Theorem 1.3. Since f^{-1} is also a quasiconformal mapping, it suffices to prove that f induces a bounded linear operator on $\dot{M}_{n/s, q}^s(\mathbb{R}^n)$, namely, if $u \in \dot{M}_{n/s, q}^s(\mathbb{R}^n)$, then $u \circ f \in \dot{M}_{n/s, q}^s(\mathbb{R}^n)$ and $\|u \circ f\|_{\dot{M}_{n/s, q}^s(\mathbb{R}^n)} \lesssim \|u\|_{\dot{M}_{n/s, q}^s(\mathbb{R}^n)}$. To this end, let $u \in \dot{M}_{n/s, q}^s(\mathbb{R}^n)$. Without loss of generality, we may assume that $\|u\|_{\dot{M}_{n/s, q}^s(\mathbb{R}^n)} = 1$. Let $\vec{g} \in \mathbb{D}^s(u)$ and $\|\vec{g}\|_{L^{n/s}(\mathbb{R}^n, \ell^q)} \leq 2$. For our convenience, by abuse of notation, we set $g_t \equiv g_k$ for all $t \in [2^{-k-1}, 2^{-k})$ and $k \in \mathbb{Z}$. Moreover, since either $J_f(x) > 0$ for almost all $x \in \mathbb{R}^n$ or $J_f(x) < 0$ for almost all $x \in \mathbb{R}^n$ (see, for example, [19, Remark 5.2]), without loss of generality, we may further assume that $J_f(x) > 0$ for almost all $x \in \mathbb{R}^n$.

Due to Theorem 2.1, the task of the proof of Theorem 1.3 is reduced to finding a suitable $\vec{h} \in \widetilde{\mathbb{D}}^{s, s, N}(u \circ f)$ with $\|\vec{h}\|_{L^{n/s}(\mathbb{R}^n, \ell^q)} \lesssim 1$ for some integer N . To this end, we consider the following three cases: (i) $q = n/s$, (ii) $q \in (n/s, \infty]$, (iii) $q \in (0, n/s)$. We pointed out that in Case (i), we only use the above basic properties of quasiconformal mappings in Propositions 5.1 and 5.2 and Lemma 5.1; in Case (ii), we need the reverse Hölder inequality; while in Case (iii), we apply Lemma 2.3 and the reverse Hölder inequality, and establish a subtle pointwise estimate via non-increasing rearrangement functions (see (5.7) below).

Case (i) $q = n/s$. In this case, by Proposition 5.2(iii), there exists $K_0 \in \mathbb{N}$ such that for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$|f(B(x, 2^{K_0}r) \setminus B(x, 2r))| \geq |f(B(x, 2r))|.$$

Then for all $x \in \mathbb{R}^n$ such that $f(x)$ is a Lebesgue point of u , and for all $k \in \mathbb{Z}$, similarly to the proof of Lemma 2.1, by Proposition 5.2(ii), we have that

$$|u \circ f(x) - u_{f(B(x, 2^{-k}))}| \leq \sum_{j \geq k} |u_{f(B(x, 2^{-j-1}))} - u_{f(B(x, 2^{-j}))}|$$

$$\begin{aligned}
&\lesssim \sum_{j \geq k} \int_{f(B(x, 2^{-j}))} |u(y) - u_{f(B(x, 2^{-j+K_0}) \setminus B(x, 2^{-j+1}))}| dy \\
&\lesssim \sum_{j \geq k} \int_{f(B(x, 2^{-j}))} \int_{f(B(x, 2^{-j+K_0}) \setminus B(x, 2^{-j+1}))} |u(y) - u(z)| dz dy,
\end{aligned}$$

where, in the penultimate inequality, we used the fact that

$$|f(B(x, 2^{-j-1}))| \sim |f(B(x, 2^{-j}))|,$$

which is obtained by Proposition 5.2(ii).

Since f is a quasisymmetric mapping, there exists $K_1 \in \mathbb{N}$ such that for all $y \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,

$$L_f(f^{-1}(y), 2^{-j+K_0+1}) \leq 2^{K_1} \ell_f(f^{-1}(y), 2^{-j}) \leq 2^{K_1} L_f(f^{-1}(y), 2^{-j}).$$

For all $k \in \mathbb{Z}$, set $\tilde{g}_k \equiv \sum_{j=k}^{k+K_1} g_j$. Then we know that $\{\tilde{g}_k\}_{k \in \mathbb{Z}} \in \mathbb{D}^{s, K_1, 0}(u)$ and, moreover,

$$\|\{\tilde{g}_k\}_{k \in \mathbb{Z}}\|_{L^{n/s}(\mathbb{R}^n, \ell^{n/s})} \lesssim \|\tilde{g}\|_{L^{n/s}(\mathbb{R}^n, \ell^{n/s})} \lesssim 1.$$

By abuse of notation, we write that $\tilde{g}_t \equiv \tilde{g}_k$ for every $t \in [2^{-k-1}, 2^{-k})$ and all $k \in \mathbb{Z}$.

For almost all $y \in f(B(x, 2^{-j}))$ and $z \in f(B(x, 2^{-j+K_0}) \setminus B(x, 2^{-j+1}))$, since

$$\ell_f(f^{-1}(y), 2^{-j}) \leq |y - z| \leq L_f(f^{-1}(y), 2^{-j+K_0+1}),$$

$$\ell_f(f^{-1}(z), 2^{-j}) \leq |y - z| \leq L_f(f^{-1}(z), 2^{-j+K_0+1})$$

and

$$|y - z| \leq |y - f(x)| + |f(x) - z| \leq 2L_f(x, 2^{-j+K_0}),$$

we have

$$\begin{aligned}
|u(y) - u(z)| &\leq |y - z|^s [g_{|y-z|}(y) + g_{|y-z|}(z)] \\
&\lesssim [L_f(x, 2^{-j+K_0})]^s [\tilde{g}_{L_f(f^{-1}(y), 2^{-j+K_0+1})}(y) + \tilde{g}_{L_f(f^{-1}(z), 2^{-j+K_0+1})}(z)],
\end{aligned}$$

which further yields that

$$|u \circ f(x) - u_{f(B(x, 2^{-k}))}| \lesssim \sum_{j \geq k-K_0-1} [L_f(x, 2^{-j})]^s \int_{f(B(x, 2^{-j}))} \tilde{g}_{L_f(f^{-1}(y), 2^{-j})}(y) dy.$$

For all $x \in \mathbb{R}^n$ and all $j \in \mathbb{Z}$, set

$$(5.2) \quad h_j(x) \equiv 2^{js} [L_f(x, 2^{-j})]^s \int_{f(B(x, 2^{-j}))} \tilde{g}_{L_f(f^{-1}(y), 2^{-j})}(y) dy.$$

Then

$$|u \circ f(x) - u_{f(B(x, 2^{-k}))}| \lesssim \sum_{j \geq k-K_0-1} 2^{-js} h_j(x).$$

Moreover, \vec{h} is a constant multiple of an element of $\widetilde{\mathbb{D}}^{s,s,K_0+2}(u \circ f)$. In fact, for every pair of Lebesgue points $x, y \in \mathbb{R}^n$ with $|x - y| \in [2^{-k-1}, 2^{-k})$, we have

$$|u \circ f(x) - u \circ f(y)| \leq |u \circ f(x) - u_{f(B(x, 2^{-k}))}| + |u \circ f(y) - u_{f(B(x, 2^{-k}))}|.$$

By Proposition 5.2(ii) and an argument similar to the above, we also have

$$\begin{aligned} & |u \circ f(y) - u_{f(B(x, 2^{-k+1}))}| \\ & \lesssim |u \circ f(y) - u_{f(B(y, 2^{-k+1}))}| + |u_{f(B(y, 2^{-k+1}))} - u_{f(B(x, 2^{-k}))}| \\ & \lesssim \sum_{j \geq k-K_0-2} 2^{-js} h_j(y) + \int_{f(B(y, 2^{-k+1}))} |u(z) - u_{f(B(y, 2^{-k+1}))}| dz \\ & \lesssim \sum_{j \geq k-K_0-2} 2^{-js} h_j(y), \end{aligned}$$

and hence

$$(5.3) \quad |u \circ f(x) - u \circ f(y)| \lesssim \sum_{j \geq k-K_0-2} 2^{-js} [h_j(x) + h_j(y)],$$

which implies that \vec{h} is a constant multiple of an element of $\widetilde{\mathbb{D}}^{s,s,K_0+2}(u \circ f)$.

Now we estimate $\|\vec{h}\|_{L^{n/s}(\mathbb{R}^n, \ell^{n/s})}$. In fact, from (ii) and (iii) of Proposition 5.2 and the fact that f is quasisymmetric, it follows that for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$,

$$(5.4) \quad [L_f(x, 2^{-j})]^n \sim |f(B(x, 2^{-j}))|,$$

which together with the Hölder inequality implies that

$$h_j(x) \lesssim 2^{js} \left\{ \int_{f(B(x, 2^{-j}))} [\widetilde{g}_{L_f(f^{-1}(y), 2^{-j})}(y)]^{n/s} dy \right\}^{s/n}.$$

Noticing that $y \in f(B(x, 2^{-j}))$ implies that $x \in B(f^{-1}(y), 2^{-j})$, by Proposition 5.2(i), we have

$$\begin{aligned} \|\vec{h}\|_{L^{n/s}(\mathbb{R}^n, \ell^{n/s})}^{n/s} & \lesssim \sum_{j \in \mathbb{Z}} 2^{jn} \int_{\mathbb{R}^n} \int_{f(B(x, 2^{-j}))} [\widetilde{g}_{L_f(f^{-1}(y), 2^{-j})}(y)]^{n/s} dy dx \\ & \lesssim \sum_{j \in \mathbb{Z}} 2^{jn} \int_{\mathbb{R}^n} [\widetilde{g}_{L_f(f^{-1}(y), 2^{-j})}(y)]^{n/s} \left\{ \int_{B(f^{-1}(y), 2^{-j})} dx \right\} dy \\ & \lesssim \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} [\widetilde{g}_{L_f(f^{-1}(y), 2^{-j})}(y)]^{n/s} dy \\ & \lesssim \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} \left(\# \left\{ j \in \mathbb{Z} : L_f(f^{-1}(y), 2^{-j}) \in [2^{-k-1}, 2^{-k}) \right\} \right) [\widetilde{g}_k(y)]^{n/s} dy, \end{aligned}$$

where $\#E$ denotes the cardinality of a set $E \subset \mathbb{Z}$. Moreover, observe that for all $k \in \mathbb{Z}$ and $y \in \mathbb{R}^n$, we have

$$(5.5) \quad \# \left\{ j \in \mathbb{Z} : L_f(f^{-1}(y), 2^{-j}) \in [2^{-k-1}, 2^{-k}) \right\} \lesssim 1.$$

Indeed, if $i, j \in \mathbb{Z}$ with $i > j$, $L_f(f^{-1}(y), 2^{-i}), L_f(f^{-1}(y), 2^{-j}) \in [2^{-k-1}, 2^{-k})$, then by (5.4) and Proposition 5.2(iii),

$$\frac{1}{2} \leq \frac{L_f(f^{-1}(y), 2^{-i})}{L_f(f^{-1}(y), 2^{-j})} \lesssim \frac{|f(B(f^{-1}(y), 2^{-i}))|^{1/n}}{|f(B(f^{-1}(y), 2^{-j}))|^{1/n}} \lesssim 2^{(j-i)\alpha/n},$$

which implies that $i - j \leq N$ for some constant N independent of i, j and y , and hence (5.5) follows. Then by (5.5), we further obtain

$$\|\vec{h}\|_{L^{n/s}(\mathbb{R}^n, \ell^{n/s})}^{n/s} \lesssim \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} [\tilde{g}_k(y)]^{n/s} dy \lesssim \|\vec{g}\|_{L^{n/s}(\mathbb{R}^n, \ell^{n/s})}^{n/s} \lesssim 1,$$

which implies that $\|u \circ f\|_{\dot{M}_{n/s, n/s}^s(\mathbb{R}^n)} \lesssim 1$. That is, Theorem 1.3 is true for the space $\dot{M}_{n/s, n/s}^s(\mathbb{R}^n)$.

Case (ii) $q \in (n/s, \infty]$. In this case, we still take $\vec{h} \equiv \{h_j\}_{j \in \mathbb{Z}}$ as a variant of the fractional s -Hajlasz gradient of $u \circ f$, where h_j is given in (5.2). Then we will control h_j by a suitable maximal function via an application of the reverse Hölder inequality satisfied by J_f . In fact, by Lemma 5.1, (5.4) and the Hölder inequality, we have

$$\begin{aligned} h_j(x) &= \left[\frac{|f(B(x, 2^{-j}))|}{|B(x, 2^{-j})|} \right]^{-1+s/n} \int_{B(x, 2^{-j})} \tilde{g}_{L_f(z, 2^{-j})}(f(z)) J_f(z) dz \\ &\lesssim \left[\frac{|f(B(x, 2^{-j}))|}{|B(x, 2^{-j})|} \right]^{-1+s/n} \left\{ \int_{B(x, 2^{-j})} \left[\tilde{g}_{L_f(z, 2^{-j})}(f(z)) \right]^p [J_f(z)]^{ps/n} dz \right\}^{1/p} \\ &\quad \times \left\{ \int_{B(x, 2^{-j})} [J_f(z)]^{p(n-s)/[n(p-1)]} dz \right\}^{(p-1)/p}, \end{aligned}$$

where we take $p \in (1, n/s)$ to be close to n/s so that $p(n-s)/n(p-1) < R_f$. Therefore, by the reverse Hölder inequality given in Proposition 5.1, and Proposition 5.2(i), we obtain

$$h_j(x) \lesssim \left\{ \mathcal{M} \left(\left[\tilde{g}_{L_f(\cdot, 2^{-j})} \circ f \right]^p [J_f]^{ps/n} \right) (x) \right\}^{1/p},$$

where we recall that \mathcal{M} denotes the Hardy-Littlewood maximal function. Therefore, the Fefferman-Stein vector-valued maximal inequality on \mathcal{M} , $p < n/s < q$, (5.5) and Lemma 5.1 yield that

$$\begin{aligned} \|\vec{h}\|_{L^{n/s}(\mathbb{R}^n, \ell^q)}^{n/s} &\lesssim \int_{\mathbb{R}^n} \left(\sum_{j \in \mathbb{Z}} \left\{ \mathcal{M} \left(\left[\tilde{g}_{L_f(\cdot, 2^{-j})} \circ f \right]^p [J_f]^{ps/n} \right) (x) \right\}^{q/p} \right)^{n/(sq)} dx \\ &\lesssim \int_{\mathbb{R}^n} \left(\sum_{j \in \mathbb{Z}} \left[\tilde{g}_{L_f(x, 2^{-j})} \circ f(x) \right]^q \right)^{n/(sq)} J_f(x) dx \end{aligned}$$

$$\begin{aligned}
&\lesssim \int_{\mathbb{R}^n} \left(\sum_{j \in \mathbb{Z}} [\tilde{g}_j \circ f(x)]^q \right)^{n/(sq)} J_f(x) dx \\
&\lesssim \int_{\mathbb{R}^n} \left(\sum_{j \in \mathbb{Z}} [\tilde{g}_j(x)]^q \right)^{n/(sq)} dx \lesssim 1.
\end{aligned}$$

Thus, Theorem 1.3 is true for the space $\dot{M}_{n/s, q}^s(\mathbb{R}^n)$ with $q \in (n/s, \infty]$.

Case (iii) $q \in (0, n/s)$. In this case, for given $q \in (n/(n+s), n/s)$, we choose $\delta \in (0, 1]$ such that

$$0 < \delta < \frac{nq(R_f - 1)}{nR_f - sq},$$

where R_f is as in (5.1) on \mathbb{R}^n . It is easy to check that

$$\frac{n - s\delta}{n} \frac{q}{q - \delta} < R_f.$$

Observe that we can take $p \in (1, q/\delta)$ and close to q/δ such that

$$(5.6) \quad \frac{n - s\delta}{n} \frac{p}{p - 1} < R_f.$$

We also let $\epsilon, \epsilon' \in (\max\{n(q-1)/q, 0\}, s)$ such that $\epsilon < \epsilon'$.

We now claim that there exists a measurable set $E \subset \mathbb{R}^n$ with $|E| = 0$ such that for all $x, y \in \mathbb{R}^n \setminus E$ with $|x - y| \in [2^{-k-1}, 2^{-k})$,

$$\begin{aligned}
(5.7) \quad |u \circ f(x) - u \circ f(y)| &\lesssim \sum_{j \geq k} \inf_{c \in \mathbb{R}} \left(\int_{B(f(x), 2L_f(x, 2^{-j}))} |u(y) - c|^\delta dy \right)^{1/\delta} \\
&\quad + \sum_{j \geq k} \inf_{c \in \mathbb{R}} \left(\int_{B(f(y), 2L_f(y, 2^{-j}))} |u(y) - c|^\delta dy \right)^{1/\delta},
\end{aligned}$$

where the implicit constant is independent of x, y, k and u , but may depend on δ .

Assume this claim holds for the moment. Observe that by Proposition 5.2(iii) and (5.4), there exists $K_2 \in \mathbb{N}$ such that for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, $4L_f(x, 2^{-j}) \leq \ell_f(x, 2^{-j+K_2})$, and hence

$$B(f(x), 4L_f(x, 2^{-j})) \subset B(f(x), \ell_f(x, 2^{-j+K_2})) \subset f(B(x, 2^{-j+K_2})).$$

Then by Lemma 2.3, we have that

$$\begin{aligned}
&\sum_{j \geq k} \inf_{c \in \mathbb{R}} \left(\int_{B(f(x), 2L_f(x, 2^{-j}))} |u(y) - c|^\delta dy \right)^{1/\delta} \\
&\lesssim \sum_{j \geq k} [L_f(x, 2^{-j})]^{\epsilon'} \sum_{i \geq -3 - \log L_f(x, 2^{-j})} 2^{-i(s-\epsilon')} \left\{ \int_{B(f(x), L_f(x, 2^{-j+K_2}))} [g_i(z)]^\delta dz \right\}^{1/\delta}.
\end{aligned}$$

Notice that by (5.6), the reverse Hölder inequality given in Proposition 5.1, the Hölder inequality, Proposition 5.2(i) and Lemma 5.1, we obtain that

$$\begin{aligned}
& \int_{f(B(x, 2^{-j+K_2}))} [g_i(z)]^\delta dz \\
& \lesssim \frac{|B(x, 2^{-j+K_2})|}{|f(B(x, 2^{-j+K_2}))|} \int_{B(x, 2^{-j+K_2})} [g_i \circ f(y)]^\delta J_f(y) dy \\
& \lesssim \frac{|B(x, 2^{-j+K_2})|}{|f(B(x, 2^{-j+K_2}))|} \left\{ \int_{B(x, 2^{-j+K_2})} [g_i \circ f(y)]^{p\delta} [J_f(y)]^{p\delta s/n} dy \right\}^{1/p} \\
& \quad \times \left\{ \int_{B(x, 2^{-j+K_2})} [J_f(y)]^{p(n-s\delta)/[n(p-1)]} dy \right\}^{(p-1)/p} \\
& \lesssim \left[\frac{|B(x, 2^{-j+K_2})|}{|f(B(x, 2^{-j+K_2}))|} \right]^{\delta s/n} \left[\mathcal{M}([g_i \circ f]^{p\delta} [J_f]^{p\delta s/n})(x) \right]^{1/p}.
\end{aligned}$$

Thus, by (5.4),

$$\begin{aligned}
& \sum_{j \geq k} \inf_{c \in \mathbb{R}} \left(\int_{B(f(x), 2L_f(x, 2^{-j}))} |u(y) - c|^\delta dy \right)^{1/\delta} \\
& \lesssim \sum_{j \geq k} [L_f(x, 2^{-j})]^{\epsilon'} \sum_{i \geq -3 - \log L_f(x, 2^{-j})} 2^{-i(s-\epsilon')} \left[\frac{|B(x, 2^{-j+K_2})|}{|f(B(x, 2^{-j+K_2}))|} \right]^{s/n} \\
& \quad \times \left[\mathcal{M}([g_i \circ f]^{p\delta} [J_f]^{p\delta s/n})(x) \right]^{1/(p\delta)} \\
& \lesssim \sum_{j \geq k} 2^{-js} \sum_{i \geq -3 - \log L_f(x, 2^{-j})} 2^{-i(s-\epsilon')} [L_f(x, 2^{-j})]^{\epsilon'-s} \\
& \quad \times \left[\mathcal{M}([g_i \circ f]^{p\delta} [J_f]^{p\delta s/n})(x) \right]^{1/(p\delta)}.
\end{aligned}$$

For all $j \in \mathbb{Z}$, set

$$h_j \equiv \sum_{i \geq -3 - \log L_f(\cdot, 2^{-j})} [L_f(\cdot, 2^{-j})]^{\epsilon'-s} 2^{-i(s-\epsilon')} \left[\mathcal{M}([g_i \circ f]^{p\delta} [J_f]^{p\delta s/n}) \right]^{1/(p\delta)}.$$

By (5.7), we know that $\vec{h} \equiv \{h_j\}_{j \in \mathbb{Z}}$ is a constant multiple of an element of $\widetilde{\mathbb{D}}^{s, s, 0}(u \circ f)$. Moreover, by (5.5), we have

$$\begin{aligned}
& \|\vec{h}\|_{L^{n/s}(\mathbb{R}^n, \ell^q)}^{n/s} \\
& \lesssim \int_{\mathbb{R}^n} \left(\sum_{j \in \mathbb{Z}} \left\{ \sum_{i \geq -3 - \log L_f(x, 2^{-j})} [L_f(x, 2^{-j})]^{\epsilon'-s} 2^{-i(s-\epsilon')} \right\} \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left[\mathcal{M}([g_i \circ f]^{p\delta} [J_f]^{p\delta s/n})(x) \right]^{1/(p\delta)} \Bigg\}^q \Bigg)^{n/(sq)} dx \\
& \lesssim \int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} \left\{ \sum_{i \geq k-3} 2^{(k-i)(s-\epsilon')} \left[\mathcal{M}([g_i \circ f]^{p\delta} [J_f]^{p\delta s/n})(x) \right]^{1/(p\delta)} \right\}^q \right)^{n/(sq)} dx.
\end{aligned}$$

When $q \in (1, n/s)$, applying the Hölder inequality, we have

$$\|\vec{h}\|_{L^{n/s}(\mathbb{R}^n, \ell^q)}^{n/s} \lesssim \int_{\mathbb{R}^n} \left(\sum_{k \in \mathbb{Z}} \sum_{i \geq k-3} 2^{(k-i)(s-\epsilon')} \left[\mathcal{M}([g_i \circ f]^{p\delta} [J_f]^{p\delta s/n})(x) \right]^{q/(p\delta)} \right)^{n/(sq)} dx,$$

which, when $q \in (n/(n+s), 1]$, still holds with $2^{(k-i)(s-\epsilon')}$ replaced by $2^{(k-i)(s-\epsilon')q}$ due to (2.2). Then, by $p < q/\delta < n/(sr)$, the Fefferman-Stein vector-valued maximal inequality on \mathcal{M} and Lemma 5.1, we obtain

$$\begin{aligned}
\|\vec{h}\|_{L^{n/s}(\mathbb{R}^n, \ell^q)}^{n/s} & \lesssim \int_{\mathbb{R}^n} \left(\sum_{i \in \mathbb{Z}} \left[\mathcal{M}([g_i \circ f]^{p\delta} [J_f]^{p\delta s/n})(x) \right]^{q/(p\delta)} \right)^{n/(sq)} dx \\
& \lesssim \int_{\mathbb{R}^n} \left(\sum_{i \in \mathbb{Z}} [g_i \circ f(x)]^q \right)^{n/(sq)} J_f(x) dx \\
& \lesssim \int_{\mathbb{R}^n} \left(\sum_{i \in \mathbb{Z}} [g_i(x)]^q \right)^{n/(sq)} dx \lesssim 1,
\end{aligned}$$

which is as desired.

Finally, we prove the above claim (5.7). For each ball B , let $m_u(B)$ be a median of u on B , namely, a real number such that

$$\max \{ |\{x \in B : f(x) > m_u(B)\}|, |\{x \in B : f(x) < m_u(B)\}| \} \leq \frac{|B|}{2}.$$

Then, as proved by Fujii [8, Lemma 2.2], there exists a measurable set $E \subset \mathbb{R}^n$ with $|E| = 0$ such that

$$u(z) = \lim_{|B| \rightarrow 0, B \ni z} m_u(B)$$

for all $z \in \mathbb{R}^n \setminus E$. Thus, for all $z \in \mathbb{R}^n \setminus E$, and every sequence $\{r_j\}_{j \geq 0}$ with $r_j \rightarrow 0$ as $j \rightarrow \infty$ and $0 < r_{j+1} \leq r_j < Nr_{j+1}$ for some fixed constant N , we have

$$\begin{aligned}
|u(z) - m_u(B(z, r_0))| & \leq \sum_{j \geq 0} |m_u(B(z, r_{j+1})) - m_u(B(z, r_j))| \\
& \leq \sum_{j \geq 0} (|m_u(B(z, r_{j+1})) - c_{B(z, r_{j+1})}| + |m_u(B(z, r_j)) - c_{B(z, r_j)}|),
\end{aligned}$$

where $c_{B(z, r_j)}$ is a real number such that

$$(5.8) \quad \oint_{B(z, r_j)} |u(w) - c_{B(z, r_j)}|^\delta dw \leq 2 \inf_{c \in \mathbb{R}} \oint_{B(z, r_j)} |u(w) - c|^\delta dw.$$

To estimate $|m_u(B(z, r_{j+1})) - c_{B(z, r_j)}|$ and $|m_u(B(z, r_j)) - c_{B(z, r_j)}|$, recall that the non-increasing rearrangement of a measurable function v is defined by

$$v^*(t) \equiv \inf\{\alpha > 0 : |\{w \in \mathbb{R}^n : |v(w)| > \alpha\}| \leq t\}.$$

Then, for every ball B and number $c \in \mathbb{R}$, obviously, we can take $m_{u-c}(B) = m_u(B) - c$ as a median of $u - c$ on B . Then, by [8, Lemma 2.1],

$$|m_u(B) - c| = |m_{u-c}(B)| \leq m_{|u-c|}(B),$$

which further implies that

$$(5.9) \quad |m_u(B) - c| \leq (|u - c|_{\chi_B})^*(|B|/2) \leq \left\{ 2 \int_B |u(w) - c|^\delta dw \right\}^{1/\delta}.$$

Indeed, letting $\sigma \equiv \int_B |u(w) - c|^\delta dw$, by Chebyshev's inequality, we have

$$\begin{aligned} \left| \left\{ w \in B : |u(w) - c| > (2\sigma)^{1/\delta} \right\} \right| &= \left| \left\{ w \in B : |u(w) - c|^\delta > 2\sigma \right\} \right| \\ &\leq (2\sigma)^{-1} \int_B |u(w) - c|^\delta dw = \frac{|B|}{2}, \end{aligned}$$

which implies the second inequality of (5.9). For the first inequality, since

$$|\{w \in B : |u(w) - c| \geq m_{|u-c|}(B)\}| = |B| - |\{w \in B : |u(w) - c| < m_{|u-c|}(B)\}| \geq \frac{|B|}{2},$$

for all $\alpha < m_{|u-c|}(B)$, we have $|\{w \in B : |u(w) - c| > \alpha\}| \geq |B|/2$, which implies that $\alpha < (|u - c|_{\chi_B})^*(|B|/2)$ and hence $m_{|u-c|}(B) \leq (|u - c|_{\chi_B})^*(|B|/2)$. This gives the first inequality of (5.9).

Combining (5.9), (5.8) and $r_{j+1} \leq r_j \leq Nr_j$ yields that

$$|m_u(B(z, r_j)) - c_{B(z, r_j)}| \lesssim \inf_{c \in \mathbb{R}} \left\{ \int_{B(z, r_j)} |u(w) - c|^\delta dw \right\}^{1/\delta}$$

and

$$\begin{aligned} (5.10) \quad |m_u(B(z, r_{j+1})) - c_{B(z, r_j)}| &\leq \left\{ 2 \int_{B(z, r_{j+1})} |u(w) - c_{B(z, r_j)}|^\delta dw \right\}^{1/\delta} \\ &\lesssim \inf_{c \in \mathbb{R}} \left\{ \int_{B(z, r_j)} |u(w) - c|^\delta dw \right\}^{1/\delta}. \end{aligned}$$

Therefore,

$$(5.11) \quad |u(z) - m_u(B(z, r_0))| \lesssim \sum_{j \geq 0} \inf_{c \in \mathbb{R}} \left\{ \int_{B(z, r_j)} |u(w) - c|^\delta dw \right\}^{1/\delta}.$$

For all $x, y \in \mathbb{R}^n \setminus E$ with $2^{-k-1} \leq |x - y| < 2^{-k}$, write

$$\begin{aligned} |u \circ f(x) - u \circ f(y)| &\leq |u \circ f(x) - m_u(B(f(x), 2L_f(x, 2^{-k})))| \\ &\quad + |m_u(B(f(y), L_f(y, 2^{-k-1}))) - m_u(B(f(x), 2L_f(x, 2^{-k})))| \\ &\quad + |u \circ f(y) - m_u(B(f(y), L_f(y, 2^{-k-1})))| \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

Since

$$L_f(z, 2^{-j+1}) \leq L_f(z, 2^{-j}) \leq 2^{K_1} L_f(z, 2^{-j+1}),$$

by (5.11), we have

$$I_1 \lesssim \sum_{j \geq 0} \inf_{c \in \mathbb{R}} \left\{ \int_{B(f(x), 2L_f(x, 2^{-j}))} |u(w) - c|^\delta dw \right\}^{1/\delta}$$

and

$$I_3 \lesssim \sum_{j \geq 1} \inf_{c \in \mathbb{R}} \left\{ \int_{B(f(y), 2L_f(y, 2^{-j}))} |u(w) - c|^\delta dw \right\}^{1/\delta}.$$

Moreover, by an argument similar to (5.10), we have

$$\begin{aligned} I_2 &\lesssim |m_u(B(f(y), L_f(y, 2^{-k-1}))) - c_{B(f(x), 2L_f(x, 2^{-k}))}| \\ &\quad + |m_u(B(f(x), 2L_f(x, 2^{-k}))) - c_{B(f(y), L_f(y, 2^{-k-1}))}| \\ &\lesssim \inf_{c \in \mathbb{R}} \left\{ \int_{B(f(x), 2L_f(x, 2^{-k}))} |u(w) - c|^\delta dw \right\}^{1/\delta}. \end{aligned}$$

Combining the estimates of I_1 , I_2 and I_3 gives the above claim (5.7). This finishes the proof of Case (iii) and hence Theorem 1.3. \square

To prove Theorem 1.4, we need the following result which is deduced from Theorem 7.11 and Corollary 7.13 of [16], [17] and the Lebesgue-Radon-Nikodym theorem.

Proposition 5.3. *Let \mathcal{X} and \mathcal{Y} be locally compact Ahlfors n -regular spaces for some $n > 1$ and assume that \mathcal{X} admits a weak $(1, n)$ -Poincaré inequality. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a quasisymmetric mapping. Then*

(i) *f is absolutely continuous, $J_f \in \mathcal{B}_r(\mathcal{X})$ for some $r \in (1, \infty]$, $J_f d\mu_{\mathcal{X}}$ is a doubling measure and*

$$\int_E J_f(x) d\mu_{\mathcal{X}}(x) = |f(E)|$$

for every measurable set $E \subset \mathcal{X}$. Here J_f denotes the volume (Radon-Nikodym) derivative of f , namely,

$$J_f(x) \equiv \lim_{r \rightarrow 0} \frac{\mu_{\mathcal{Y}}(f(B(x, r)))}{\mu_{\mathcal{X}}(B(x, r))},$$

which exists and is finite for almost all $x \in \mathcal{X}$.

(ii) *f^{-1} is also a quasisymmetric mapping, absolutely continuous and for almost all $x \in \mathcal{X}$, $J_{f^{-1}}(f(x)) = [J_f(x)]^{-1}$.*

Proof of Theorem 1.4. With the assumptions of Theorem 1.4, by Corollary 4.8 and Theorem 5.7 of [16], we know that f is actually an η -quasisymmetric mapping for some homeomorphism $\eta : [0, \infty) \rightarrow [0, \infty)$. Moreover, since the assumptions of Theorem 1.4 also imply those of Proposition 5.3, we have that $J_f \in \mathcal{B}_r(\mathcal{X})$ and $J_f d\mu_{\mathcal{X}}$ is also a doubling measure, which together with [26] imply that J_f is a weight in the sense of Muckenhoupt. Therefore, a variant of Proposition 5.2(iii) still holds in this setting. Moreover, recall that by [15, Proposition 10.8], for every pair of sets A and B satisfying $A \subset B \subset \mathcal{X}$ and $0 < \text{diam } A \leq \text{diam } B < \infty$,

$$(5.12) \quad \frac{1}{2\eta\left(\frac{\text{diam } B}{\text{diam } A}\right)} \leq \frac{\text{diam } f(A)}{\text{diam } f(B)} \leq \eta\left(\frac{2 \text{diam } A}{\text{diam } B}\right).$$

Then, with the aid of these facts, Proposition 5.3 and Proposition 4.1, the proof of Theorem 1.4 is essentially the same as that of Theorem 1.3. We omit the details. \square

As the above proof shows, with the assumptions of Proposition 5.3, a similar conclusion of Theorem 1.4 still holds.

Corollary 5.1. *Let the assumptions be as in Proposition 5.3. Then for all $s \in (0, 1]$ and $q \in (0, \infty]$, a quasisymmetric mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$ induces an equivalence between $\dot{M}_{n/s, q}^s(\mathcal{X})$ and $\dot{M}_{n/s, q}^s(\mathcal{Y})$.*

Finally, we turn to the proof of Theorem 5.1. To this end, we need the following Lebesgue theorem for Hajlasz-Sobolev functions, which is proved by modifying the proof of [13, Theorem 4.4] slightly (see also [18, Theorem 4.4]).

Lemma 5.2. *Let \mathcal{X} be an Ahlfors n -regular space with $n > 0$ and $s \in (0, n)$. Then for every $u \in \dot{B}_{n/s}^s(\mathcal{X})$, there exists a set F such that $\mathcal{X} \setminus F$ has Hausdorff dimension zero and $\tilde{u}(x) \equiv \lim_{r \rightarrow 0} \int_{B(x, r)} u(z) d\mu(z)$ exists for all $x \in F$.*

Proof. Let $u \in \mathcal{B}_{n/s}^s(\mathcal{X}) \subset \dot{M}^{s, n/s}(\mathcal{X})$. For all $x \in \mathcal{X}$, define

$$\tilde{u}(x) \equiv \limsup_{r \rightarrow 0} \int_{B(x, r)} u(z) d\mu(z).$$

By Lemma 2.2, u is locally integrable and hence, $\tilde{u}(x) = u(x)$ for almost all $x \in \mathcal{X}$. Denote by F the set of all $x \in \mathcal{X}$ such that $\tilde{u}(x) = \lim_{r \rightarrow 0} \int_{B(x, r)} u(z) d\mu(z)$. Then, to show Lemma 5.2, it suffices to prove that $\mathcal{X} \setminus F$ has Hausdorff dimension zero. Let $g \in \mathcal{D}^s(\mathcal{X}) \cap L^{n/s}(\mathcal{X})$ and $\epsilon \in (0, s)$. Notice that for $x \in \mathcal{X}$ and $j, k \in \mathbb{Z}$ with $k \geq j + 1$,

$$\begin{aligned} |u_{B(x, 2^{-k})} - u_{B(x, 2^{-j})}| &\leq \sum_{i=j}^{k-1} |u_{B(x, 2^{-i-1})} - u_{B(x, 2^{-i})}| \\ &\leq \sum_{i=j}^{k-1} \int_{B(x, 2^{-i})} |u(z) - u_{B(x, 2^{-i})}| d\mu(z) \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=j}^{k-1} 2^{-is} \int_{B(x, 2^{-i})} g(z) d\mu(z) \\
&\lesssim 2^{-j(s-\epsilon)} \sup_{i \geq j} 2^{-i\epsilon} \int_{B(x, 2^{-i})} g(z) d\mu(z).
\end{aligned}$$

If $\sup_{i \geq 0} 2^{-i\epsilon} \int_{B(x, 2^{-i})} g(z) dz < \infty$, then $x \in F$. Thus,

$$\mathcal{X} \setminus F \subset G \equiv \left\{ x \in \mathcal{X} : \sup_{i \geq 0} 2^{-i\epsilon} \int_{B(x, 2^{-i})} g(z) d\mu(z) = \infty \right\}.$$

Since for all $i \in \mathbb{N}$ and $x \in \mathcal{X}$, by the Hölder inequality,

$$2^{-i\epsilon} \int_{B(x, 2^{-i})} g(z) d\mu(z) \leq \left\{ 2^{-in\epsilon/s} \int_{B(x, 2^{-i})} [g(z)]^{n/s} d\mu(z) \right\}^{s/n},$$

then by [13, Lemma 2.6], for all $N \in \mathbb{N}$, we further have

$$\begin{aligned}
&\mathcal{H}_{\infty}^{n(1-\epsilon/s)}(G \cap B(y, 1)) \\
&\leq \mathcal{H}_{\infty}^{n(1-\epsilon/s)} \left(\left\{ x \in B(y, 1) : \sup_{i \geq 0} 2^{-in\epsilon/s} \int_{B(x, 2^{-i})} [g(z)]^{n/s} d\mu(z) > N^{n/s} \right\} \right) \\
&\lesssim N^{-n/s} \int_{\mathcal{X}} [g(z)]^{n/s} d\mu(z),
\end{aligned}$$

which implies that $\mathcal{H}_{\infty}^{n(1-\epsilon/s)}(G \cap B(y, 1)) = 0$ and hence $\mathcal{H}^{n(1-\epsilon/s)}(G \cap B(y, 1)) = 0$. Here $\mathcal{H}_{\infty}^{n(1-\epsilon/s)}$ and $\mathcal{H}^{n(1-\epsilon/s)}$ denote the Hausdorff content and the Hausdorff measure, respectively. Because we are free to choose $\epsilon \in (0, s)$, we conclude that G and hence $\mathcal{X} \setminus F$ have Hausdorff dimension zero. This finishes the proof of Lemma 5.2. \square

Proof of Theorem 5.1. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be an η -quasisymmetric mapping. By [15, Proposition 10.6], f^{-1} is an $\tilde{\eta}$ -quasisymmetric with $\tilde{\eta}(t) = 1/\eta^{-1}(t^{-1})$ for all $t \in (0, \infty)$. Moreover, since \mathcal{X} and \mathcal{Y} are Ahlfors regular spaces and hence uniformly perfect, by [15, Corollary 11.5], f and f^{-1} are Hölder continuous on bounded sets of \mathcal{X} and \mathcal{Y} , respectively.

Let $u \in \dot{M}_{n_2/s_2, n_2/s_2}^{s_2}(\mathcal{Y})$ and \tilde{u} be as in Lemma 5.2. Then Lemma 5.2 says that $\tilde{u}(x) = u(x)$ for almost all $x \in \mathcal{Y}$, and the complement of the set of all the Lebesgue points of \tilde{u} is contained in $\mathcal{Y} \setminus F$ and has Hausdorff dimension zero. By abuse of notation, we still denote \tilde{u} by u . Since f^{-1} is Hölder continuous on every bounded set of \mathcal{Y} , it is easy to check that $f^{-1}(\mathcal{Y} \setminus F)$ has Hausdorff dimension zero and hence $\mu_{\mathcal{X}}(f^{-1}(\mathcal{Y} \setminus F)) = 0$. Let $\vec{g} \in \mathbb{D}^{s_2}(u)$ such that $\|\vec{g}\|_{L^{n_2/s_2}(\mathcal{Y}, \ell^{n_2/s_2})} \lesssim \|u\|_{\dot{B}_{n_2/s_2}^{s_2}(\mathcal{Y})}$ and (1.1) holds with some set E having measure zero. For all $x \in f^{-1}(F) \subset \mathcal{X}$ and for all $j \in \mathbb{Z}$, if $2^{-j} < 2 \operatorname{diam} \mathcal{X}$, set

$$h_j(x) \equiv 2^{js_1} [L_f(x, 2^{-j})]^{s_2} \int_{f(B(x, 2^{-j}))} \tilde{g}_{L_f(f^{-1}(y), 2^{-j})}(y) d\mu_{\mathcal{Y}}(y),$$

and if $2^{-j} \geq 2 \operatorname{diam} \mathcal{X}$, set $h_j \equiv 0$. Since $\mu_{\mathcal{X}}(\mathcal{X} \setminus f^{-1}(F)) = \mu_{\mathcal{X}}(f^{-1}(\mathcal{Y} \setminus F)) = 0$, \vec{h} is well-defined. Moreover, for each $x \in f^{-1}(F)$, since $f(x)$ is a Lebesgue point of u , it follows that $u_{f(B(x, 2^{-j}))} \rightarrow u \circ f(x)$ as $j \rightarrow \infty$. Observing that $\mu_{\mathcal{X}}(\mathcal{X} \setminus f^{-1}(F)) = 0$, by an argument as in the proof of Theorem 1.3, we can prove that $\vec{h} \equiv \{h_j\}_{j \in \mathbb{Z}}$ is a constant multiple of an element of $\mathbb{D}^{s_1, s_1, K_0}(u \circ f)$ for some constant K_0 determined by (5.12) and the constants appearing in (1.2) for $\mu_{\mathcal{Y}}$.

Now we estimate $\|\vec{h}\|_{L^{n_1/s_1}(\mathcal{X}, \ell^{n_1/s_1})}$. In fact, since $[L_f(x, 2^{-j})]^{n_2} \sim |f(B(x, 2^{-j}))|$, by the Hölder inequality and $n_1/s_1 = n_2/s_2$, we then have

$$h_j(x) \lesssim 2^{js_1} \left\{ \int_{f(B(x, 2^{-j}))} [\tilde{g}_{L_f(f^{-1}(y), 2^{-j})}(y)]^{n_1/s_1} d\mu_{\mathcal{Y}}(y) \right\}^{s_1/n_1}.$$

Noticing that $y \in f(B(x, 2^{-j}))$ implies that $x \in B(f^{-1}(y), 2^{-j})$, by an argument similar to that of the proof of Theorem 1.3, we have

$$\begin{aligned} & \|\vec{h}\|_{L^{n_1/s_1}(\mathcal{X}, \ell^{n_1/s_1})}^{n_1/s_1} \\ & \lesssim \int_{\mathcal{Y}} \sum_{k \in \mathbb{Z}} \left(\# \left\{ j \in \mathbb{Z} : L_f(f^{-1}(y), 2^{-j}) \in [2^{-k-1}, 2^{-k}] \right\} \right) [\tilde{g}_k(y)]^{n_1/s_1} d\mu_{\mathcal{Y}}(y). \end{aligned}$$

Moreover, observe that for all $k \in \mathbb{Z}$ and $y \in \mathcal{Y}$, by (5.12),

$$\# \left\{ j \in \mathbb{Z} : L_f(f^{-1}(y), 2^{-j}) \in [2^{-k-1}, 2^{-k}] \right\} \lesssim 1.$$

By $n_1/s_1 = n_2/s_2$, we then have

$$\|\vec{h}\|_{L^{n_1/s_1}(\mathcal{X}, \ell^{n_1/s_1})}^{n_1/s_1} \lesssim \sum_{k \in \mathbb{Z}} \int_{\mathcal{Y}} [\tilde{g}_k(y)]^{n_1/s_1} d\mu(y) \lesssim \|\vec{g}\|_{L^{n_2/s_2}(\mathcal{Y}, \ell^{n_2/s_2})}^{n_2/s_2}.$$

Thus, $\|u \circ f\|_{\dot{M}_{n_1/s_1, n_1/s_1}^{s_1}(\mathcal{X})} \lesssim \|u\|_{\dot{M}_{n_2/s_2, n_2/s_2}^{s_2}(\mathcal{Y})}$. Applying the above result to f^{-1} , we obtain that $\|u \circ f^{-1}\|_{\dot{M}_{n_1/s_1, n_1/s_1}^{s_1}(\mathcal{Y})} \lesssim \|u\|_{\dot{M}_{n_2/s_2, n_2/s_2}^{s_2}(\mathcal{X})}$, which completes the proof of Theorem 5.1. \square

Moreover, combining the proofs of Theorems 5.1 and 1.3, one can further obtain the following conclusion.

Corollary 5.2. *Let \mathcal{X} and \mathcal{Y} be Ahlfors n_1 -regular and n_2 -regular spaces with $n_1, n_2 \in (0, \infty)$, respectively. Let f be a quasisymmetric mapping from \mathcal{X} onto \mathcal{Y} , and assume that f and f^{-1} are absolutely continuous and $J_f \in \mathcal{B}_r(\mathcal{X})$ for some $r \in (1, \infty]$. Let $s_i \in (0, n_i)$ with $i = 1, 2$ satisfy $n_1/s_1 = n_2/s_2$, and $q \in (0, \infty]$. Then f induces an equivalence between $\dot{M}_{n_1/s_1, q}^{s_1}(\mathcal{X})$ and $\dot{M}_{n_2/s_2, q}^{s_2}(\mathcal{Y})$.*

In Corollary 5.2, f acts a composition operator. Moreover, with the assumptions of Corollary 5.2, by Lebesgue-Radon-Nykodym Theorem and [26], we have that $J_{f^{-1}}(y) = [J_f(f^{-1}(y))]^{-1}$ for almost all $y \in \mathcal{Y}$, and hence $J_{f^{-1}} \in \mathcal{B}_{r'}(\mathcal{Y})$ for some $r' \in (1, \infty]$.

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